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ORIGINAL PAPER

The 3D incompressible Navier–Stokes equations with partial hyperdissipation

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Abstract
The three-dimensional incompressible Navier–Stokes equations with the hyperdissipation $(-\Delta)^\gamma$ always possess global smooth solutions when $\gamma \geq \frac{5}{4}$. Tao [6] and Barbato, Morandin and Romito [1] made logarithmic reductions in the dissipation and still obtained the global regularity. This paper makes a different type of reduction in the dissipation and proves the global existence and uniqueness in the H^1-functional setting.

KEYWORDS
3D Navier–Stokes equations, fractional partial dissipation, global regularity

MSC (2010)
35Q35, 76D03

1 | INTRODUCTION

Whether or not reasonably smooth solutions of the 3D incompressible Navier–Stokes equations can blow up in finite time remains an outstanding open problem. This problem is supercritical in the sense that the standard Laplacian dissipation in the 3D Navier–Stokes equations may not be sufficient in controlling the nonlinearity and the hyperdissipative Navier–Stokes equations

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nu(-\Delta)^\alpha u = -\nabla p, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

with sufficiently smooth u_0 always possess a unique global smooth solution when $\alpha \geq \alpha_0 = \frac{5}{4}$ (see, e.g., [2,4,7]). Here the fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform (see, e.g., [5])

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi),$$

where $\widehat{f}$ denotes the standard Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^3} e^{-i\,x\cdot\xi} f(x)\,dx.$$

The critical exponent $\alpha_0 = \frac{5}{4}$ makes the energy of (1.1) invariant under the natural scaling. If (u, p) solves (1.1), then (u_λ, p_λ) with $\lambda > 0$ defined by

$$u_\lambda(x,t) = \lambda^{2\alpha-1} u\big(\lambda x, \lambda^{2\alpha} t\big), \qquad p_\lambda(x,t) = \lambda^{4\alpha-2} p\big(\lambda x, \lambda^{2\alpha} t\big)$$

also solves (1.1) with the initial data $u_{0\lambda}(x) = \lambda^{2\alpha-1} u_0(\lambda x)$. The energy associated with u_λ

$$E(u_\lambda) \equiv \sup_{t\ge 0} \int_{\mathbb{R}^3} |u_\lambda(t)|^2\, dx + \int_0^\infty \int_{\mathbb{R}^3} |\Lambda^\alpha u_\lambda|^2\, dx\, dt = \lambda^{4\alpha-5} E(u)$$

remains invariant, namely $E(u_\lambda) = E(u)$ when $\alpha = \alpha_0 = \frac{5}{4}$.

It is natural to ask if the global regularity remains valid when $\alpha < \alpha_0 = \frac{5}{4}$? Tao examined the hyperdissipative Navier–Stokes equations involving general Fourier multiplier operators [6]. A special consequence is a logarithmic reduction in the dissipation. More precisely, he shows that replacing $(-\Delta)^{\frac{5}{4}} u$ by $\frac{(-\Delta)^{\frac{5}{4}}}{\log^{1/2}(I-\Delta)} u$ still leads to a unique global solution. Tao's result was later improved by [1] to allow additional $-\frac{1}{2}$ power in the logarithm, namely (1.1) with $\frac{(-\Delta)^{\frac{5}{4}}}{\log(I-\Delta)} u$. Efforts have also been devoted to the incompressible magnetohydrodynamic equations with hyperdissipation and the results are not completely parallel (see, e.g., [8,9]).

This paper explores a different type of reduction in the dissipation. We remove some components of the hyperdissipation in (1.1). More precisely, we study the following 3D Navier–Stokes equations with fractional partial dissipation,

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 = -\partial_1 p - \nu\left(\Lambda_1^{\frac{5}{2}} + \Lambda_2^{\frac{5}{2}}\right) u_1, \\ \partial_t u_2 + (u\cdot\nabla)u_2 = -\partial_2 p - \nu\left(\Lambda_2^{\frac{5}{2}} + \Lambda_3^{\frac{5}{2}}\right) u_2, \\ \partial_t u_3 + (u\cdot\nabla)u_3 = -\partial_3 p - \nu\left(\Lambda_1^{\frac{5}{2}} + \Lambda_3^{\frac{5}{2}}\right) u_3, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.2}$$

where the fractional partial dissipation operators Λ_k^γ with $\gamma > 0$ and $k = 1,2,3$ are defined via the Fourier transform

$$\widehat{\Lambda_k^\gamma f}(\xi) = |\xi_k|^\gamma\, \hat{f}(\xi).$$

We remark that the study of the system in (1.2) is our original idea and has never been done before. We are able to show that, for any $u_0 \in H^1(\mathbb{R}^3)$, (1.2) always possesses a unique global solution. More precisely, the following theorem holds.

Theorem 1.1. *Assume $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$. Then (1.2) has a unique global solution u satisfying*

$$u \in L^\infty\big(0,\infty; H^1(\mathbb{R}^3)\big);$$

$$\left(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}\right) u_1 \in L^2\big(\mathbb{R}^3\times(0,\infty)\big), \qquad \left(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}\right) \nabla u_1 \in L^2\big(\mathbb{R}^3\times(0,\infty)\big),$$

$$\left(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}\right) u_2 \in L^2\big(\mathbb{R}^3\times(0,\infty)\big), \qquad \left(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}\right) \nabla u_2 \in L^2\big(\mathbb{R}^3\times(0,\infty)\big),$$

$$\left(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}\right) u_3 \in L^2\big(\mathbb{R}^3\times(0,\infty)\big), \qquad \left(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}\right) \nabla u_3 \in L^2\big(\mathbb{R}^3\times(0,\infty)\big).$$

In addition, the H^1-norm of u is bounded uniformly in time.

It is clear that the global existence and regularity still holds when $\Lambda_k^{\frac{5}{2}}$ $(k = 1,2,3)$ in (1.2) is replaced by $\Lambda_k^{2\gamma}$ with any $\gamma > \frac{5}{4}$. The proof for the global regularity part boils down to showing the global H^1 a priori bound. It is not clear at a first glance if one can still control the nonlinearity suitably when one directional dissipation is missing in each component of (1.2). The global L^2-bound follows directly from a standard energy estimate. The proof of the global H^1-bound is more delicate and the key is to how to effectively make use of the reduced dissipation to bound the nonlinear terms. To control the terms associated with the nonlinearity

$$I \equiv \int \nabla((u\cdot\nabla)u)\cdot\nabla u\,dx,$$

we use $\nabla\cdot u = 0$ and decompose the terms into three groups,

$$I = \sum_{i,j=1}^{3}\int \partial_i u_j\,\partial_j u_1\,\partial_i u_1\,dx + \sum_{i,j=1}^{3}\int \partial_i u_j\,\partial_j u_2\,\partial_i u_2\,dx + \sum_{i,j=1}^{3}\int \partial_i u_j\,\partial_j u_3\,\partial_i u_3\,dx.$$

The first group of terms are associated with the estimate of ∇u_1, the second group with ∇u_2 and the third group with ∇u_3. It suffices to estimate the terms in the first group and they are bounded one by one. Several tools are employed to facilitate the estimates including one-dimensional Sobolev type inequalities. Due to the lack of the fractional partial dissipation $\Lambda_3^{\frac{5}{2}}u_1$, $\Lambda_1^{\frac{5}{2}}u_2$ and $\Lambda_2^{\frac{5}{2}}u_3$, the terms containing one or more of the terms $\partial_3 u_1$, $\partial_1 u_2$ and $\partial_2 u_3$ are more difficult. Integration by parts and the divergence-free condition $\nabla\cdot u = 0$ are repeatedly applied to rebalance the derivatives.

The rest of this paper is divided into two sections. Section 2 provides the global H^1 a priori bound while Section 3 proves the uniqueness of the H^1 solutions.

2 | GLOBAL H^1-BOUND

The proof for the global existence part of Theorem 1.1 boils down to global *a priori* bounds on the solutions. This section establishes a uniform global H^1-bound on solutions of (1.2). More precisely, we prove the following proposition.

Proposition 2.1. *Assume $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$. Let u be the corresponding solution of* (1.2). *Then, the following uniform global H^1 bound holds, for any $t > 0$,*

$$\|\nabla u(t)\|_{L^2}^2 + \nu\int_0^t \left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\nabla u_1\right\|_{L^2}^2 d\tau + \nu\int_0^t \left\|\left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\nabla u_2\right\|_{L^2}^2 d\tau + \nu\int_0^t \left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\nabla u_3\right\|_{L^2}^2 d\tau$$
$$\le \|\nabla u_0\|_{L^2}^2\, e^{C\|u_0\|_{L^2}^4(1+\|u_0\|_{L^2})},$$

where, for the sake of brevity, we have written

$$\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\nabla u_1\right\|_{L^2} \equiv \left\|\Lambda_1^{\frac{5}{4}}\nabla u_1\right\|_{L^2} + \left\|\Lambda_2^{\frac{5}{4}}\nabla u_1\right\|_{L^2}.$$

A necessary step in the proof of Proposition 2.1 is the following global L^2-bound, which follows from a direct L^2 energy estimate involving (1.2).

Lemma 2.2. *Assume $u_0 \in H^1$ with $\nabla\cdot u_0 = 0$. Let u be the corresponding solution of* (1.2). *Then, for any $t \ge 0$,*

$$\|u(t)\|_{L^2}^2 + 2\nu\int_0^t\left(\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)u_1(\tau)\right\|_{L^2}^2 + \left\|\left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)u_2(\tau)\right\|_{L^2}^2 + \left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)u_3(\tau)\right\|_{L^2}^2\right)d\tau = \|u_0\|_{L^2}^2.$$

In addition, to prove Proposition 2.1, we need several tool lemmas. The first one is a Sobolev embedding inequality involving one-dimensional functions.

Lemma 2.3. *Let $2 \le p \le \infty$. Let $s > \frac{1}{2} - \frac{1}{p}$. Then, there exists a constant $C = C(p,s)$ such that, for any 1D functions $f \in H^s(\mathbb{R})$,*

$$\|f\|_{L^p(\mathbb{R})} \le C\,\|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}\,\|\Lambda^s f\|_{L^2(\mathbb{R})}^{\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

Proof of Lemma 2.3. This lemma can be proven via the Littlewood–Paley decomposition and Besov space techniques. We identify L^2 with the homogeneous Besov space $\dot{B}^s_{2,2}$ and recall the embedding relations, for any $2 \le p \le \infty$,

$$L^p \hookrightarrow \dot{B}^0_{p,2}.$$

Then, by Bernstein's inequality and Hölder's inequality,

$$\begin{aligned}
\|f\|_{L^p(\mathbb{R})} &\le C\|f\|_{\dot{B}^0_{p,2}(\mathbb{R})} = C\left[\sum_{j=-\infty}^{\infty}\|\Delta_j f\|^2_{L^p(\mathbb{R})}\right]^{\frac{1}{2}}\\
&\le C\left[\sum_{j=-\infty}^{\infty}2^{2j(\frac{1}{2}-\frac{1}{p})}\|\Delta_j f\|^2_{L^2(\mathbb{R})}\right]^{\frac{1}{2}}\\
&= C\left[\sum_{j=-\infty}^{\infty}\|\Delta_j f\|_{L^2(\mathbb{R})}^{2\left(1-\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)\right)}\left(2^{sj}\|\Delta_j f\|_{L^2(\mathbb{R})}\right)^{2\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}\right]^{\frac{1}{2}}\\
&\le C\left[\sum_{j=-\infty}^{\infty}\|\Delta_j f\|^2_{L^2(\mathbb{R})}\right]^{\frac{1}{2}\left(1-\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)\right)}\left[\sum_{j=-\infty}^{\infty}2^{2sj}\|\Delta_j f\|^2_{L^2(\mathbb{R})}\right]^{\frac{1}{2}\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}\\
&= C\|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}\|\Lambda^s f\|_{L^2(\mathbb{R})}^{\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}.
\end{aligned}$$

This completes the proof of Lemma 2.3. □

Let $J = (I-\Delta)^{\frac{1}{2}}$ denote the inhomogeneous differentiation operator. We recall two well-known calculus inequalities (see, e.g., [3, p. 334]).

Lemma 2.4. *Let $s > 0$. Let $p, p_1, p_3 \in (1,\infty)$ and $p_2, p_4 \in [1,\infty]$ satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then, for two constants C_1 and C_2,

$$\begin{aligned}
&\|J^s(f\,g)\|_{L^p} \le C_1\left(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|J^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}\right),\\
&\|J^s(f\,g) - f\,J^s g\|_{L^p} \le C_2\left(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|J^{s-1} g\|_{L^{p_3}}\|\nabla f\|_{L^{p_4}}\right).
\end{aligned}$$

The next tool lemma states one version of the Minkowski inequality, which is the foundation for exchanging two Lebesgue norms.

Lemma 2.5. *Let $f = f(x,y)$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ be a measurable function on $\mathbb{R}^m \times \mathbb{R}^n$. Let $1 \le q \le p \le \infty$. Then*

$$\Big\|\|f\|_{L^q_y(\mathbb{R}^n)}\Big\|_{L^p_x(\mathbb{R}^m)} \le \Big\|\|f\|_{L^p_x(\mathbb{R}^m)}\Big\|_{L^q_y(\mathbb{R}^n)}.$$

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Taking the L^2-inner product of ∇u with the gradient of (1.2) and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2} + \nu\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\nabla u_1\right\|^2_{L^2} + \nu\left\|\left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\nabla u_2\right\|^2_{L^2} + \nu\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\nabla u_3\right\|^2_{L^2}\\
&= -\sum_{i,j=1}^{3}\int \partial_i u_j\,\partial_j u_1\,\partial_i u_1\,dx - \sum_{i,j=1}^{3}\int \partial_i u_j\,\partial_j u_2\,\partial_i u_2\,dx - \sum_{i,j=1}^{3}\int \partial_i u_j\,\partial_j u_3\,\partial_i u_3\,dx\\
&:= I_1 + I_2 + I_3. \qquad (2.1)
\end{aligned}$$

We first estimate I_1. To do so, we write out the nine terms explicitly,

$$I_1 = -\int \Big((\partial_1 u_1)^3 + \partial_1 u_1\, \partial_1 u_2\, \partial_2 u_1 + \partial_1 u_1\, \partial_1 u_3\, \partial_3 u_1 + (\partial_2 u_1)^2\, \partial_1 u_1 + (\partial_2 u_1)^2\, \partial_2 u_2 + \partial_2 u_1\, \partial_2 u_3\, \partial_3 u_1$$
$$+ (\partial_3 u_1)^2\, \partial_1 u_1 + \partial_3 u_1\, \partial_3 u_2\, \partial_2 u_1 + (\partial_3 u_1)^2\, \partial_3 u_3 \Big)\, dx. \tag{2.2}$$

When we estimate the terms above, we keep in mind that we have the space and time L^2 integrability of the terms

$$\Lambda_1^{\frac{5}{4}} u_1, \quad \Lambda_2^{\frac{5}{4}} u_1, \quad \Lambda_2^{\frac{5}{4}} u_2, \quad \Lambda_3^{\frac{5}{4}} u_2, \quad \Lambda_1^{\frac{5}{4}} u_3, \quad \Lambda_3^{\frac{5}{4}} u_3$$

and the left hand side of (2.1) allows us to control the space and time L^2-norm of the terms

$$\Big(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}\Big)(\partial_1, \partial_2, \partial_3) u_1, \quad \Big(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}\Big)(\partial_1, \partial_2, \partial_3) u_2, \quad \Big(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}\Big)(\partial_1, \partial_2, \partial_3) u_3.$$

The terms in (2.2) will be labelled as $I_{11}, I_{12}, \ldots$ according to the order they appear in (2.2). Throughout the rest of this proof, we use $\|f\|_{L^2}$ to denote $\|f\|_{L^2(\mathbb{R}^3)}$ and use $\|f\|_{L^2_{x_1}}$ and $\|f\|_{L^2_{x_1x_2}}$ to denote the one-dimensional L^2-norm (in terms of x_1) and the two-dimensional L^2-norm (in terms of x_1 and x_2), respectively.

We first deal with I_{12}, the second term in (2.2). We will return to I_{11} later. Integration by parts yields

$$\begin{aligned} I_{12} &= -\int \partial_1 u_1\, \partial_1 u_2\, \partial_2 u_1\, dx = \int u_2\, \partial_1\partial_2 u_1\, \partial_1 u_1\, dx + \int u_2\, \partial_1\partial_1 u_1\, \partial_2 u_1\, dx \\ &:= I_{121} + I_{122}. \end{aligned}$$

By Hölder's inequality and Minkowski's inequality in Lemma 2.5,

$$\begin{aligned} |I_{121}| &\le \|\partial_1\partial_2 u_1\|_{L^2_{x_3} L^4_{x_2} L^2_{x_1}} \|\partial_1 u_1\|_{L^\infty_{x_3} L^2_{x_2} L^4_{x_1}} \|u_2\|_{L^2_{x_3} L^4_{x_2} L^4_{x_1}} \\ &\le \|\partial_1\partial_2 u_1\|_{L^2_{x_3} L^2_{x_1} L^4_{x_2}} \|\partial_1 u_1\|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} \|u_2\|_{L^2_{x_3} L^4_{x_2} L^4_{x_1}} \\ &\le \|\partial_1\partial_2 u_1\|_{L^2_{x_1x_3} L^4_{x_2}} \|\partial_1 u_1\|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} \|u_2\|_{L^2_{x_3} L^4_{x_1x_2}} \end{aligned}$$

where we have written

$$\|f\|_{L^2_{x_1x_3} L^4_{x_2}} = \Big\| \|f\|_{L^4_{x_2}} \Big\|_{L^2_{x_1x_3}}, \quad \|f\|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} = \Big\| \Big\| \|f\|_{L^4_{x_1}} \Big\|_{L^\infty_{x_3}} \Big\|_{L^2_{x_2}}.$$

By Sobolev's and Gagliardo–Nirenberg's inequalities,

$$\|\partial_1\partial_2 u_1\|_{L^2_{x_1x_3} L^4_{x_2}} = \Big\| \|\partial_1\partial_2 u_1\|_{L^4_{x_2}} \Big\|_{L^2_{x_1x_3}} \le C \Big\| \Big\| \Lambda_2^{\frac{1}{4}} \partial_1\partial_2 u_1 \Big\|_{L^2_{x_2}} \Big\|_{L^2_{x_1x_3}} = C \Big\| \Lambda_2^{\frac{1}{4}} \partial_1\partial_2 u_1 \Big\|_{L^2}$$

and

$$\|u_2\|_{L^2_{x_3} L^4_{x_1x_2}} = \Big\| \|u_2\|_{L^4_{x_1x_2}} \Big\|_{L^2_{x_3}} \le \Big\| \|u_2\|^{\frac{1}{2}}_{L^2_{x_1x_2}} \|\nabla_h u_2\|^{\frac{1}{2}}_{L^2_{x_1x_2}} \Big\|_{L^2_{x_3}} \le C\, \|u_2\|^{\frac{1}{2}}_{L^2} \|\nabla_h u_2\|^{\frac{1}{2}}_{L^2},$$

where $\nabla_h = (\partial_1, \partial_2)$. Applying Gagliardo–Nirenberg's inequality and Lemma 2.3 with $p = \infty$ and $s = 1$, we have

$$\|\partial_1 u_1\|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} \le C \Big\| \Lambda_1^{\frac{1}{4}} \partial_1 u_1 \Big\|_{L^2_{x_1x_2} L^\infty_{x_3}} \le C \Big\| \Lambda_1^{\frac{5}{4}} u_1 \Big\|^{\frac{1}{2}}_{L^2} \Big\| \Lambda_3 \Lambda_1^{\frac{5}{4}} u_1 \Big\|^{\frac{1}{2}}_{L^2}.$$

Combining the estimates above and applying Young's inequality yield

$$
\begin{aligned}
|I_{121}| &\le C\left\|\Lambda_2^{\frac14}\partial_1\partial_2 u_1\right\|_{L^2}\|u_2\|_{L^2}^{\frac12}\|\nabla_h u_2\|_{L^2}^{\frac12}\left\|\Lambda_1^{\frac54}u_1\right\|_{L^2}^{\frac12}\left\|\Lambda_3\Lambda_1^{\frac54}u_1\right\|_{L^2}^{\frac12}\\
&\le \frac{\nu}{128}\left\|\Lambda_2^{\frac54}\partial_1 u_1\right\|_{L^2}^2+\frac{\nu}{128}\left\|\Lambda_1^{\frac54}\partial_3 u_1\right\|_{L^2}^2+C\,\|u_2\|_{L^2}^2\left\|\Lambda_1^{\frac54}u_1\right\|_{L^2}^2\|\nabla_h u_2\|_{L^2}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_{122}| &\le C\,\|\partial_1\partial_1 u_1\|_{L^2_{x_1x_2}L^4_{x_2}}\|u_2\|_{L^2_{x_3}L^4_{x_1x_2}}\|\partial_2 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\\
&\le C\left\|\Lambda_2^{\frac14}\partial_1\partial_1 u_1\right\|_{L^2}\|u_2\|_{L^2}^{\frac12}\|\nabla_h u_2\|_{L^2}^{\frac12}\left\|\Lambda_1^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12}\left\|\Lambda_3\Lambda_1^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12}.
\end{aligned}
$$

Due to the elementary inequalities

$$
|\xi_2|^{\frac14}|\xi_1|\le\frac45|\xi_1|^{\frac54}+\frac15|\xi_2|^{\frac54},\qquad |\xi_1|^{\frac14}|\xi_2|\le\frac15|\xi_1|^{\frac54}+\frac45|\xi_2|^{\frac54}
$$

and Plancherel's theorem, we have

$$
\begin{aligned}
\left\|\Lambda_2^{\frac14}\partial_1\partial_1 u_1\right\|_{L^2}&\le C\Big(\|\Lambda_1^{\frac54}\partial_1 u_1\|_{L^2}+\|\Lambda_2^{\frac54}\partial_1 u_1\|_{L^2}\Big),\\
\left\|\Lambda_1^{\frac14}\partial_2 u_1\right\|_{L^2}&\le C\Big(\|\Lambda_1^{\frac54}u_1\|_{L^2}+\|\Lambda_2^{\frac54}u_1\|_{L^2}\Big).
\end{aligned}
$$

Therefore, by Young's inequality,

$$
|I_{122}|\le\frac{\nu}{128}\left\|\Big(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\Big)\partial_1 u_1\right\|_{L^2}^2+\frac{\nu}{128}\left\|\Big(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\Big)\partial_3 u_1\right\|_{L^2}^2+C\,\|u_2\|_{L^2}^2\left\|\Big(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\Big)u_1\right\|_{L^2}^2\|\nabla_h u_2\|_{L^2}^2.
$$

We return to estimate I_{11}, the first term in (2.1). I_{11} can be handled similarly as I_{12}. Integrating by parts and then bounding it as I_{121}, we have

$$
\begin{aligned}
I_{11}&=2\int u_1\,\partial_1\partial_1 u_1\,\partial_1 u_1\,dx\\
&\le 2\,\|\partial_1\partial_1 u_1\|_{L^2_{x_1x_3}L^4_{x_2}}\|u_1\|_{L^2_{x_3}L^4_{x_1x_2}}\|\partial_1 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\\
&\le C\left\|\Lambda_2^{\frac14}\partial_1\partial_1 u_1\right\|_{L^2}\|u_1\|_{L^2}^{\frac12}\|\nabla_h u_1\|_{L^2}^{\frac12}\left\|\Lambda_1^{\frac14}\partial_1 u_1\right\|_{L^2}^{\frac12}\left\|\Lambda_3\Lambda_1^{\frac14}\partial_1 u_1\right\|_{L^2}^{\frac12}\\
&\le\frac{\nu}{128}\left\|\Lambda_2^{\frac14}\partial_1\partial_1 u_1\right\|_{L^2}^2+\frac{\nu}{128}\left\|\Lambda_1^{\frac54}\partial_3 u_1\right\|_{L^2}^2+C\,\|u_1\|_{L^2}^2\left\|\Lambda_1^{\frac54}u_1\right\|_{L^2}^2\|\nabla_h u_1\|_{L^2}^2\\
&\le\frac{\nu}{64}\left\|\Big(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\Big)\nabla u_1\right\|_{L^2}^2+C\,\|u_1\|_{L^2}^2\left\|\Lambda_1^{\frac54}u_1\right\|_{L^2}^2\|\nabla_h u_1\|_{L^2}^2.
\end{aligned}
$$

We now estimate I_{13}. Integrating by parts and invoking the divergence-free condition $\nabla\cdot u=0$, we have

$$
\begin{aligned}
I_{13}\;&=\;-\int\partial_1 u_1\,\partial_1 u_3\,\partial_3 u_1\,dx=\int u_1\,\partial_3\partial_1 u_1\,\partial_1 u_3\,dx+\int u_1\,\partial_1 u_1\,\partial_1\partial_3 u_3\,dx\\
&=\int u_1\,\partial_3\partial_1 u_1\,\partial_1 u_3\,dx-\int u_1\,\partial_1 u_1\,\partial_1\partial_1 u_1\,dx-\int u_1\,\partial_1 u_1\,\partial_1\partial_2 u_2\,dx\\
&:=I_{131}+I_{132}+I_{133}.
\end{aligned}
$$

I_{132} is the same as I_{11} and I_{133} obeys a similar bound as I_{121},

$$|I_{133}| \le \|\partial_1\partial_2 u_2\|_{L^2_{x_1x_3}L^4_{x_2}} \|u_1\|_{L^2_{x_3}L^4_{x_1x_2}} \|\partial_1 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}$$

$$\le C\left\|\Lambda_2^{\frac14}\partial_1\partial_2 u_2\right\|_{L^2} \|u_1\|_{L^2}^{\frac12}\|\nabla_h u_1\|_{L^2}^{\frac12}\left\|\Lambda_1^{\frac14}\partial_1 u_1\right\|_{L^2}^{\frac12}\left\|\Lambda_3\Lambda_1^{\frac14}\partial_1 u_1\right\|_{L^2}^{\frac12}$$

$$\le \frac{\nu}{128}\left\|\Lambda_2^{\frac54}\partial_1 u_2\right\|_{L^2}^2 + \frac{\nu}{128}\left\|\Lambda_1^{\frac54}\partial_3 u_1\right\|_{L^2}^2 + C\,\|u_1\|_{L^2}^2\left\|\Lambda_1^{\frac54}u_1\right\|_{L^2}^2\|\nabla_h u_1\|_{L^2}^2.$$

It suffices to deal with I_{131}, which can be handled as I_{121}. In fact, as in the estimates of I_{121},

$$I_{131} \le C\,\|\partial_3\partial_1 u_1\|_{L^2_{x_1x_3}L^4_{x_2}} \|\partial_1 u_3\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}} \|u_1\|_{L^2_{x_3}L^4_{x_1x_2}}$$

$$\le C\left\|\Lambda_2^{\frac14}\partial_3\partial_1 u_1\right\|_{L^2}\left\|\Lambda_1^{\frac14}\partial_1 u_3\right\|_{L^2}^{\frac12}\left\|\Lambda_3\Lambda_1^{\frac14}\partial_1 u_3\right\|_{L^2}^{\frac12}\|u_1\|_{L^2}^{\frac12}\|\nabla_h u_1\|_{L^2}^{\frac12}$$

$$\le C\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)\partial_3 u_1\right\|_{L^2}\left\|\Lambda_1^{\frac54}u_3\right\|_{L^2}^{\frac12}\left\|\Lambda_1^{\frac54}\partial_3 u_3\right\|_{L^2}^{\frac12}\|u_1\|_{L^2}^{\frac12}\|\nabla_h u_1\|_{L^2}^{\frac12}$$

$$\le \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)\partial_3 u_1\right\|_{L^2}^2 + \frac{\nu}{128}\left\|\Lambda_1^{\frac54}\partial_3 u_3\right\|_{L^2}^2 + C\,\|u_1\|_{L^2}^2\left\|\Lambda_1^{\frac54}u_3\right\|_{L^2}^2\|\nabla_h u_1\|_{L^2}^2.$$

This settles the estimate of I_{13}. We turn to I_{14} and I_{15} and they both can be handled as I_{12}. By integration by parts,

$$I_{14} = 2\int u_1\,\partial_2 u_1\,\partial_1\partial_2 u_1, \qquad I_{15} = 2\int u_2\,\partial_2 u_1\,\partial_2\partial_2 u_1.$$

Going through a similar procedure as in the estimates of I_{12}, we obtain

$$|I_{14}| \le 2\,\|\partial_1\partial_2 u_1\|_{L^2_{x_2x_3}L^4_{x_1}} \|u_1\|_{L^2_{x_3}L^4_{x_1x_2}} \|\partial_2 u_1\|_{L^2_{x_1}L^\infty_{x_3}L^4_{x_2}}$$

$$\le C\left\|\Lambda_1^{\frac14}\partial_1\partial_2 u_1\right\|_{L^2}\|u_1\|_{L^2}^{\frac12}\|\nabla_h u_1\|_{L^2}^{\frac12}\left\|\Lambda_2^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12}\left\|\Lambda_3\Lambda_2^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12}$$

$$\le \frac{\nu}{128}\left\|\Lambda_1^{\frac54}\partial_2 u_1\right\|_{L^2}^2 + \frac{\nu}{128}\left\|\Lambda_2^{\frac54}\partial_3 u_1\right\|_{L^2}^2 + C\|u_1\|_{L^2}^2\left\|\Lambda_2^{\frac54}u_1\right\|_{L^2}^2\|\nabla_h u_1\|_{L^2}^2$$

$$\le \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)\nabla u_1\right\|_{L^2}^2 + C\,\|u_1\|_{L^2}^2\left\|\Lambda_2^{\frac54}u_1\right\|_{L^2}^2\|\nabla_h u_1\|_{L^2}^2$$

and

$$|I_{15}| \le 2\,\|\partial_2\partial_2 u_1\|_{L^2_{x_2x_3}L^4_{x_1}} \|u_2\|_{L^2_{x_3}L^4_{x_1x_2}} \|\partial_2 u_1\|_{L^2_{x_1}L^\infty_{x_3}L^4_{x_2}}$$

$$\le C\|\Lambda_1\partial_2\partial_2 u_1\|_{L^2}\|u_2\|_{L^2}^{\frac12}\|\nabla_h u_2\|_{L^2}^{\frac12}\left\|\Lambda_2^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12}\left\|\Lambda_3\Lambda_2^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12}$$

$$\le \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)\partial_2 u_1\right\|_{L^2}^2 + \frac{\nu}{128}\left\|\Lambda_2^{\frac54}\partial_3 u_1\right\|_{L^2}^2 + C\,\|u_2\|_{L^2}^2\left\|\Lambda_2^{\frac54}u_1\|_{L^2}^2\right\|\nabla_h u_2\|_{L^2}^2$$

$$\le \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)\nabla u_1\right\|_{L^2}^2 + C\,\|u_2\|_{L^2}^2\left\|\Lambda_2^{\frac54}u_1\|_{L^2}^2\right\|\nabla_h u_2\|_{L^2}^2.$$

The estimates of I_{16} is more delicate. The integrand of I_{16},

$$I_{16} = -\int \partial_2 u_3\, \partial_3 u_1\, \partial_2 u_1\, dx$$

involves $\partial_3 u_1$ and $\partial_2 u_3$, but the first component equation of (1.2) involves no dissipation in the third direction and the third component equation involves no dissipation in the second direction. By integration by parts twice,

$$\begin{aligned} I_{16} &= \int u_1\, \partial_3\partial_2 u_3\, \partial_2 u_1\, dx + \int u_1\, \partial_2 u_3\, \partial_3\partial_2 u_1\, dx \\ &= \int u_1\, \partial_3\partial_2 u_3\, \partial_2 u_1\, dx - \int u_3\, \partial_2 u_1\, \partial_3\partial_2 u_1\, dx - \int u_1 u_3\, \partial_2\partial_3\partial_2 u_1\, dx \\ &:= I_{161} + I_{162} + I_{163}. \end{aligned}$$

The first two terms I_{161} and I_{162} can be estimated similarly as I_{12}. In fact,

$$\begin{aligned} I_{161} &\le \|\partial_3\partial_2 u_3\|_{L^2_{x_1x_2} L^4_{x_3}} \|\partial_2 u_1\|_{L^2_{x_3} L^\infty_{x_1} L^4_{x_2}} \|u_1\|_{L^2_{x_1} L^4_{x_2x_3}} \\ &\le C \left\|\Lambda_3^{\frac14} \partial_3\partial_2 u_3\right\|_{L^2} \left\|\Lambda_2^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12} \left\|\Lambda_1\Lambda_2^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12} \|u_1\|_{L^2}^{\frac12} \|(\partial_2,\partial_3)u_1\|_{L^2}^{\frac12} \\ &\le \frac{\nu}{128}\left\|\Lambda_3^{\frac54}\partial_2 u_3\right\|_{L^2}^2 + \frac{\nu}{128}\left\|\Lambda_2^{\frac54}\partial_1 u_1\right\|_{L^2}^2 + C\,\|u_1\|_{L^2}^2 \left\|\Lambda_2^{\frac54} u_1\right\|_{L^2}^2 \|(\partial_2,\partial_3)u_1\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} I_{162} &\le \|\partial_3\partial_2 u_1\|_{L^2_{x_1x_3} L^4_{x_2}} \|\partial_2 u_1\|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} \|u_3\|_{L^2_{x_3} L^4_{x_1x_2}} \\ &\le C \left\|\Lambda_2^{\frac14}\partial_3\partial_2 u_1\right\|_{L^2} \left\|\Lambda_1^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12} \left\|\Lambda_3\Lambda_1^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12} \|u_3\|_{L^2}^{\frac12} \|\nabla_h u_3\|_{L^2}^{\frac12} \\ &\le \frac{\nu}{128}\left\|\Lambda_2^{\frac54}\partial_3 u_1\right\|_{L^2}^2 + \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)\partial_3 u_1\right\|_{L^2}^2 + C\,\|u_3\|_{L^2}^2 \left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)u_1\right\|_{L^2}^2 \|(\partial_1,\partial_2)u_3\|_{L^2}^2. \end{aligned}$$

I_{163} is estimated differently. By Hölder's inequality and the boundedness of singular integral operators, especially Riesz transforms, on L^2,

$$\begin{aligned} |I_{163}| &\le \int \left|\Lambda_2^{\frac34}(u_1u_3)\,\Lambda_2^{-1}\partial_2\Lambda_2^{\frac14}\partial_2\partial_3 u_1\right| \\ &\le C\left\|\Lambda_2^{\frac34}(u_1u_3)\right\|_{L^2}\left\|\left(\Lambda_2^{-1}\partial_2\right)\Lambda_2^{\frac14}\partial_2\partial_3 u_1\right\|_{L^2} \\ &\le \left\|\Lambda_2^{\frac34}(u_1u_3)\right\|_{L^2}\left\|\Lambda_2^{\frac14}\partial_2\partial_3 u_1\right\|_{L^2}, \end{aligned} \tag{2.3}$$

where we have used the fact that the Riesz transform $\Lambda_2^{-1}\partial_2$ is bounded on L^2. By Lemma 2.4,

$$\left\|\Lambda_2^{\frac34}(u_1u_3)\right\|_{L^2_{x_2}} \le C\left(\|\Lambda_2^{\frac34}u_1\|_{L^4_{x_2}}\|u_3\|_{L^4_{x_2}} + \|u_1\|_{L^4_{x_2}}\|\Lambda_2^{\frac34}u_3\|_{L^4_{x_2}}\right)$$

and thus, by Höler's inequality and Lemma 2.3,

$$\begin{aligned}\left\|\Lambda_2^{\frac{3}{4}}(u_1u_3)\right\|_{L^2} &\le C\left(\left\|\Lambda_2^{\frac{3}{4}}u_1\right\|_{L^\infty_{x_3}L^4_{x_1x_2}}\|u_3\|_{L^2_{x_3}L^4_{x_1x_2}}+\left\|\Lambda_2^{\frac{3}{4}}u_3\right\|_{L^\infty_{x_3}L^4_{x_1x_2}}\|u_1\|_{L^2_{x_3}L^4_{x_1x_2}}\right)\\
&\le C\left(\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2u_1\right\|_{L^2_{x_1x_2}L^\infty_{x_3}}\|u_3\|_{L^2}^{\frac{1}{2}}\|(\partial_1,\partial_2)u_3\|_{L^2}^{\frac{1}{2}}+\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2u_3\right\|_{L^2_{x_1x_2}L^\infty_{x_3}}\|u_1\|_{L^2}^{\frac{3}{5}}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)u_1\right\|_{L^2}^{\frac{2}{5}}\right)\\
&\le C\left(\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2u_1\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2\partial_3u_1\right\|_{L^2}^{\frac{1}{2}}\|u_3\|_{L^2}^{\frac{1}{2}}\|(\partial_1,\partial_2)u_3\|_{L^2}^{\frac{1}{2}}\right.\\
&\quad\left.+\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2u_3\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2\Lambda_3u_3\right\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2}^{\frac{3}{5}}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)u_1\right\|_{L^2}^{\frac{2}{5}}\right).\end{aligned}\tag{2.4}$$

Due to the lack of the second component dissipation in the equation of u_3, or the lack of L^2-integrability in time of $\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2u_3\right\|_{L^2}$, we need to further interpolate this term,

$$\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2u_3\right\|_{L^2_{x_1}}\le\|\Lambda_2u_3\|_{L^2_{x_1}}^{\frac{4}{5}}\left\|\Lambda_1^{\frac{5}{4}}\Lambda_2u_3\right\|_{L^2_{x_1}}^{\frac{1}{5}}$$

and thus

$$\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2u_3\right\|_{L^2}\le\|\Lambda_2u_3\|_{L^2}^{\frac{4}{5}}\left\|\Lambda_1^{\frac{5}{4}}\Lambda_2u_3\right\|_{L^2}^{\frac{1}{5}}.\tag{2.5}$$

Inserting (2.5) in (2.4) yields,

$$\begin{aligned}\left\|\Lambda_2^{\frac{3}{4}}(u_1u_3)\right\|_{L^2}&\le C\left(\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2u_1\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2\partial_3u_1\right\|_{L^2}^{\frac{1}{2}}\|u_3\|_{L^2}^{\frac{1}{2}}\|(\partial_1,\partial_2)u_3\|_{L^2}^{\frac{1}{2}}\right.\\
&\quad\left.+\|\Lambda_2u_3\|_{L^2}^{\frac{2}{5}}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\partial_2u_3\right\|_{L^2}^{\frac{3}{5}}\|u_1\|_{L^2}^{\frac{3}{5}}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)u_1\right\|_{L^2}^{\frac{2}{5}}\right).\end{aligned}\tag{2.6}$$

Combining (2.3) and (2.6) and applying Young's inequality, we have

$$\begin{aligned}|I_{163}|\le{}&\frac{\nu}{128}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\partial_3u_1\right\|_{L^2}^2+C\,\|u_3\|_{L^2}^2\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)u_1\right\|_{L^2}^2\|(\partial_1,\partial_2)u_3\|_{L^2}^2\\
&+\frac{\nu}{128}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\partial_2u_3\right\|_{L^2}^2+C\,\|u_1\|_{L^2}^3\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)u_1\right\|_{L^2}^2\|\Lambda_2u_3\|_{L^2}^2.\end{aligned}$$

We now turn to the last three terms in (2.2). Due to $\nabla\cdot u=0$, the last three terms in (2.2) can regrouped into two terms,

$$-\int\partial_3u_1\,\partial_3u_2\,\partial_2u_1+\int\partial_2u_2(\partial_3u_1)^2\,dx:=I_{17}+I_{18}.$$

I_{17} can be estimated as I_{12}. By integration by parts,

$$I_{17}=\int u_1\,\partial_3\partial_3u_2\,\partial_2u_1\,dx+\int u_1\,\partial_3\partial_2u_1\,\partial_3u_2\,dx:=I_{171}+I_{172}.$$

I_{171} can be estimated similarly as I_{12} and we have

$$
\begin{aligned}
|I_{171}| &\le \|\partial_3\partial_3 u_2\|_{L^2_{x_1x_3}L^4_{x_2}} \|\partial_2 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}} \|u_1\|_{L^2_{x_3}L^4_{x_1x_2}} \\
&\le C\left\|\Lambda_2^{\frac14}\partial_3\partial_3 u_2\right\|_{L^2}\left\|\Lambda_1^{\frac14}\partial_2 u_1\right\|_{L^2}^{\frac12}\left\|\Lambda_1^{\frac14}\partial_2\partial_3 u_1\right\|_{L^2}^{\frac12}\|u_1\|_{L^2}^{\frac12}\|(\partial_1,\partial_2)u_1\|_{L^2}^{\frac12} \\
&\le \frac{\nu}{128}\left\|\left(\Lambda_2^{\frac54},\Lambda_3^{\frac54}\right)\partial_3 u_2\right\|_{L^2}^2 + \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)\partial_3 u_1\right\|_{L^2}^2 + C\,\|u_1\|_{L^2}^2\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)u_1\right\|_{L^2}^2\|(\partial_1,\partial_2)u_1\|_{L^2}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_{172}| &\le \|\partial_3\partial_2 u_1\|_{L^2_{x_2x_3}L^4_{x_1}} \|\partial_3 u_2\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_2}} \|u_1\|_{L^2_{x_3}L^4_{x_1x_2}} \\
&\le C\left\|\Lambda_1^{\frac14}\partial_3\partial_2 u_1\right\|_{L^2}\left\|\Lambda_2^{\frac14}\partial_3 u_2\right\|_{L^2}^{\frac12}\left\|\Lambda_3\Lambda_2^{\frac14}\partial_3 u_2\right\|_{L^2}^{\frac12}\|u_1\|_{L^2}^{\frac12}\|\nabla_h u_1\|_{L^2}^{\frac12} \\
&\le \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)\partial_3 u_1\right\|_{L^2}^2 + \frac{\nu}{128}\left\|\left(\Lambda_2^{\frac54},\Lambda_3^{\frac54}\right)\partial_3 u_2\right\|_{L^2}^2 + C\,\|u_1\|_{L^2}^2\left\|\left(\Lambda_2^{\frac54},\Lambda_3^{\frac54}\right)u_2\right\|_{L^2}^2\|(\partial_1,\partial_2)u_1\|_{L^2}^2.
\end{aligned}
$$

Finally we deal with I_{18},

$$
I_{18} = \int \partial_2 u_2(\partial_3 u_1)^2\,dx.
$$

Due to the appearance of $(\partial_3 u_1)^2$ and the lack of dissipation in the third direction in the equation of u_1, the handling of this term is more delicate. By integration by parts,

$$
I_{18} = -2\int u_2\,\partial_3 u_1\,\partial_2\partial_3 u_1\,dx.
$$

By Hölder's inequality and by Minkowski's inequality,

$$
|I_{18}| \le 2\,\|\partial_2\partial_3 u_1\|_{L^2_{x_1x_3}L^4_{x_2}}\|u_2\|_{L^2_{x_1x_2}L^\infty_{x_3}}\|\partial_3 u_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}}. \tag{2.7}
$$

By Lemma 2.3 and an interpolation inequality,

$$
\begin{aligned}
\|\partial_3 u_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}} &\le C\left\|\Lambda_2^{\frac14}\partial_3 u_1\right\|_{L^2_{x_2x_3}L^\infty_{x_1}} \\
&\le C\left\|\Lambda_2^{\frac14}\partial_3 u_1\right\|_{L^2}^{\frac12}\left\|\Lambda_2^{\frac14}\Lambda_1\partial_3 u_1\right\|_{L^2}^{\frac12} \\
&\le C\,\|\partial_3 u_1\|_{L^2}^{\frac25}\left\|\Lambda_2^{\frac54}\partial_3 u_1\right\|_{L^2}^{\frac1{10}}\left\|\Lambda_2^{\frac14}\Lambda_1\partial_3 u_1\right\|_{L^2}^{\frac12} \\
&\le C\,\|\partial_3 u_1\|_{L^2}^{\frac25}\left\|\left(\Lambda_1^{\frac54},\Lambda_2^{\frac54}\right)\partial_3 u_1\right\|_{L^2}^{\frac35},
\end{aligned} \tag{2.8}
$$

where we have invoked the interpolation inequality

$$
\left\|\Lambda_2^{\frac14}\partial_3 u_1\right\|_{L^2_{x_2}} \le C\,\|\partial_3 u_1\|_{L^2_{x_2}}^{\frac45}\left\|\Lambda_2^{\frac54}\partial_3 u_1\right\|_{L^2_{x_2}}^{\frac15}.
$$

In addition,

$$\|u_2\|_{L^2_{x_1x_2}L^\infty_{x_3}} \le C\,\|u_2\|_{L^2}^{\frac{3}{5}}\left\|\Lambda_3^{\frac{5}{4}}u_2\right\|_{L^2}^{\frac{2}{5}}. \tag{2.9}$$

Inserting (2.8) and (2.9) in (2.7) yields

$$|I_{18}| \le C\left\|\Lambda_2^{\frac{5}{4}}\partial_3 u_1\right\|_{L^2}\|u_2\|_{L^2}^{\frac{3}{5}}\left\|\Lambda_3^{\frac{5}{4}}u_2\right\|_{L^2}^{\frac{2}{5}}\|\partial_3 u_1\|_{L^2}^{\frac{2}{5}}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\partial_3 u_1\right\|_{L^2}^{\frac{3}{5}} \le \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\partial_3 u_1\right\|_{L^2}^2 + C\,\|u_2\|_{L^2}^3\left\|\Lambda_3^{\frac{5}{4}}u_2\right\|_{L^2}^2\|\partial_3 u_1\|_{L^2}^2.$$

We remark that I_{16} and I_{18} could have been treated in a similar fashion. We intentionally estimated them differently to make available different approaches that serve the same purpose. We have finished estimating all terms in I_1 in (2.1). I_2 and I_3 in (2.1) can be similarly estimated as the terms in I_1 and we omit the details. Putting all these estimates together, we have

$$\begin{aligned}
&\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\nabla u_1\right\|_{L^2}^2 + \nu\left\|\left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\nabla u_2\right\|_{L^2}^2 + \nu\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\nabla u_3\right\|_{L^2}^2\\
&\le C\left(\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)u_1\right\|_{L^2}^2 + \left\|\left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)u_2\right\|_{L^2}^2 + \left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)u_3\right\|_{L^2}^2\right)\left(\|u\|_{L^2}^2 + \|u\|_{L^2}^3\right)\|\nabla u\|_{L^2}^2.
\end{aligned}$$

Gronwall's inequality then yields the desired global bound. This completes the proof of Proposition 2.1. □

3 | UNIQUENESS

This section proves the uniqueness part of Theorem 1.1. In fact, we prove a proposition that is slightly stronger than the desired uniqueness. The uniqueness in the following proposition does not require both solutions are in the regularity class induced by the existence part.

Proposition 3.1. *Let $T > 0$. Assume that $u^{(1)}$ and $u^{(2)}$ are two solutions of* (1.2) *satisfying,*

$$\begin{aligned}
&u^{(i)} \in L^\infty\left(0,T;H^1\left(\mathbb{R}^3\right)\right) \quad \text{for } i=1,2,\\
&\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)u_1^{(2)} \in L^2\left(\mathbb{R}^3\times(0,T)\right), \qquad \left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\nabla u_1^{(2)} \in L^2\left(\mathbb{R}^3\times(0,T)\right),\\
&\left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)u_2^{(2)} \in L^2\left(\mathbb{R}^3\times(0,T)\right), \qquad \left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\nabla u_2^{(2)} \in L^2\left(\mathbb{R}^3\times(0,T)\right),\\
&\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)u_3^{(2)} \in L^2\left(\mathbb{R}^3\times(0,T)\right), \qquad \left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\nabla u_3^{(2)} \in L^2\left(\mathbb{R}^3\times(0,T)\right).
\end{aligned}$$

Then $u^{(1)} = u^{(2)}$ on $\mathbb{R}^3\times(0,T)$.

Proof. Let $p^{(1)}$ and $p^{(2)}$ be the pressures associated with $u^{(1)}$ and $u^{(2)}$, respectively. Then the differences $\widetilde{u} = u^{(1)} - u^{(2)}$ and $\widetilde{p} = p^{(1)} - p^{(2)}$ satisfy

$$\begin{cases}
\partial_t\widetilde{u}_1 + \left(u^{(1)}\cdot\nabla\right)\widetilde{u}_1 + (\widetilde{u}\cdot\nabla)u_1^{(2)} = -\partial_1\widetilde{p} - \nu\left(\Lambda_1^{\frac{5}{2}} + \Lambda_2^{\frac{5}{2}}\right)\widetilde{u}_1,\\
\partial_t\widetilde{u}_2 + \left(u^{(1)}\cdot\nabla\right)\widetilde{u}_2 + (\widetilde{u}\cdot\nabla)u_2^{(2)} = -\partial_2\widetilde{p} - \nu\left(\Lambda_2^{\frac{5}{2}} + \Lambda_3^{\frac{5}{2}}\right)\widetilde{u}_2,\\
\partial_t\widetilde{u}_3 + \left(u^{(1)}\cdot\nabla\right)\widetilde{u}_3 + (\widetilde{u}\cdot\nabla)u_3^{(2)} = -\partial_3\widetilde{p} - \nu\left(\Lambda_1^{\frac{5}{2}} + \Lambda_3^{\frac{5}{2}}\right)\widetilde{u}_3,\\
\nabla\cdot\widetilde{u} = 0,\\
\widetilde{u}(x,0) = \widetilde{u}_0(x).
\end{cases} \tag{3.1}$$

Dotting (3.1) with $\widetilde{u}$ and invoking the divergence-free conditions $\nabla \cdot u^{(1)} = 0$ and $\nabla \cdot \widetilde{u} = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|_{L^2}^2 + \nu\left\|\left(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}\right)\widetilde{u}_1\right\|_{L^2}^2 + \nu\left\|\left(\Lambda_2^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}\right)\widetilde{u}_2\right\|_{L^2}^2 + \nu\left\|\left(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}\right)\widetilde{u}_3\right\|_{L^2}^2 \\
&= -\int (\widetilde{u}\cdot\nabla)u^{(2)}\cdot\widetilde{u}\,dx \\
&= -\int (\widetilde{u}\cdot\nabla)u_1^{(2)}\widetilde{u}_1\,dx - \int (\widetilde{u}\cdot\nabla)u_2^{(2)}\widetilde{u}_2\,dx - \int (\widetilde{u}\cdot\nabla)u_3^{(2)}\widetilde{u}_3\,dx \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

We estimate J_1 and write its terms explicitly,

$$\begin{aligned}
J_1 &= -\int \widetilde{u}_1\partial_1 u_1^{(2)}\widetilde{u}_1\,dx - \int \widetilde{u}_2\partial_2 u_1^{(2)}\widetilde{u}_1\,dx - \int \widetilde{u}_3\partial_3 u_1^{(2)}\widetilde{u}_1\,dx \\
&=: J_{11} + J_{12} + J_{13}.
\end{aligned}$$

By Hölder's inequality and Lemma 2.3,

$$\begin{aligned}
|J_{11}| &\le \|\widetilde{u}_1\|_{L^2}\|\widetilde{u}_1\|_{L^2_{x_3}L^\infty_{x_2}L^4_{x_1}}\left\|\partial_1 u_1^{(2)}\right\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}} \\
&\le C\|\widetilde{u}_1\|_{L^2}\left\|\Lambda_1^{\frac{1}{4}}\widetilde{u}_1\right\|_{L^2_{x_1x_3}L^\infty_{x_2}}\left\|\Lambda_1^{\frac{1}{4}}\partial_1 u_1^{(2)}\right\|_{L^2_{x_1x_2}L^\infty_{x_3}} \\
&\le C\|\widetilde{u}_1\|_{L^2}\left\|\Lambda_1^{\frac{1}{4}}\widetilde{u}_1\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_1^{\frac{1}{4}}\Lambda_2\widetilde{u}_1\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_1^{\frac{1}{4}}\partial_1 u_1^{(2)}\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_1^{\frac{1}{4}}\Lambda_3\partial_1 u_1^{(2)}\right\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Inserting the interpolation inequality above,

$$\left\|\Lambda_1^{\frac{1}{4}}\widetilde{u}_1\right\|_{L^2_{x_1}} \le C\|\widetilde{u}_1\|_{L^2_{x_1}}^{\frac{4}{5}}\left\|\Lambda_1^{\frac{5}{4}}\widetilde{u}_1\right\|_{L^2_{x_1}}^{\frac{1}{5}},$$

we obtain

$$\begin{aligned}
|J_{11}| &\le C\|\widetilde{u}_1\|_{L^2}^{\frac{7}{5}}\left\|\left(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}\right)\widetilde{u}_1\right\|_{L^2}^{\frac{3}{5}}\left\|\Lambda_1^{\frac{5}{4}}u_1^{(2)}\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_1^{\frac{5}{4}}\Lambda_3 u_1^{(2)}\right\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}\right)\widetilde{u}_1\right\|_{L^2}^2 + C\|\widetilde{u}_1\|_{L^2}^2\left\|\Lambda_1^{\frac{5}{4}}u_1^{(2)}\right\|_{L^2}^{\frac{5}{7}}\left\|\Lambda_1^{\frac{5}{4}}\Lambda_3 u_1^{(2)}\right\|_{L^2}^{\frac{5}{7}}.
\end{aligned}$$

By Hölder's inequality and Lemma 2.3,

$$\begin{aligned}
|J_{12}| &\le \left\|\partial_2 u_1^{(2)}\right\|_{L^2}\|\widetilde{u}_2\|_{L^2_{x_1}L^\infty_{x_3}L^4_{x_2}}\|\widetilde{u}_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}} \\
&\le C\left\|\partial_2 u_1^{(2)}\right\|_{L^2}\left\|\Lambda_2^{\frac{1}{4}}\widetilde{u}_2\right\|_{L^2_{x_1x_2}L^\infty_{x_3}}\left\|\Lambda_2^{\frac{1}{4}}\widetilde{u}_1\right\|_{L^2_{x_2x_3}L^\infty_{x_1}} \\
&\le C\left\|\partial_2 u_1^{(2)}\right\|_{L^2}\left\|\Lambda_2^{\frac{1}{4}}\widetilde{u}_2\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_2^{\frac{1}{4}}\partial_3\widetilde{u}_2\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_2^{\frac{1}{4}}\widetilde{u}_1\right\|_{L^2}^{\frac{1}{2}}\left\|\Lambda_2^{\frac{1}{4}}\partial_1\widetilde{u}_1\right\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Invoking the interpolation inequalities

$$\left\|\Lambda_2^{\frac{1}{4}}\widetilde{u}_2\right\|_{L^2_{x_2}} \le C\,\|\widetilde{u}_2\|^{\frac{4}{5}}_{L^2_{x_2}}\left\|\Lambda_2^{\frac{5}{4}}\widetilde{u}_2\right\|^{\frac{1}{5}}_{L^2_{x_2}}, \qquad \left\|\Lambda_2^{\frac{1}{4}}\widetilde{u}_1\right\|_{L^2_{x_2}} \le C\,\|\widetilde{u}_1\|^{\frac{4}{5}}_{L^2_{x_2}}\left\|\Lambda_2^{\frac{5}{4}}\widetilde{u}_1\right\|^{\frac{1}{5}}_{L^2_{x_2}}$$

and thus

$$\left\|\Lambda_2^{\frac{1}{4}}\widetilde{u}_2\right\|_{L^2} \le C\,\|\widetilde{u}_2\|^{\frac{4}{5}}_{L^2}\left\|\Lambda_2^{\frac{5}{4}}\widetilde{u}_2\right\|^{\frac{1}{5}}_{L^2}, \qquad \left\|\Lambda_2^{\frac{1}{4}}\widetilde{u}_1\right\|_{L^2} \le C\,\|\widetilde{u}_1\|^{\frac{4}{5}}_{L^2}\left\|\Lambda_2^{\frac{5}{4}}\widetilde{u}_1\right\|^{\frac{1}{5}}_{L^2},$$

we have

$$\begin{aligned}|J_{12}| &\le C\left\|\partial_2 u_1^{(2)}\right\|_{L^2}\|\widetilde{u}_1\|^{\frac{2}{5}}_{L^2}\|\widetilde{u}_2\|^{\frac{2}{5}}_{L^2}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\widetilde{u}_1\right\|^{\frac{3}{5}}_{L^2}\left\|\left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\widetilde{u}_2\right\|^{\frac{3}{5}}_{L^2}\\ &\le \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\widetilde{u}_1\right\|^2_{L^2}+\frac{\nu}{128}\left\|\left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\widetilde{u}_2\right\|^2_{L^2}+C\left\|\partial_2 u_1^{(2)}\right\|^{\frac{5}{2}}_{L^2}\|\widetilde{u}_1\|_{L^2}\,\|\widetilde{u}_2\|_{L^2}.\end{aligned}$$

The estimate for J_{13} is similar to that for J_{12}. In fact,

$$\begin{aligned}|J_{13}| &\le \left\|\partial_3 u_1^{(2)}\right\|_{L^2}\|\widetilde{u}_1\|_{L^2_{x_3}L^\infty_{x_2}L^4_{x_1}}\|\widetilde{u}_3\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\\ &\le \frac{\nu}{128}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\widetilde{u}_1\right\|^2_{L^2}+\frac{\nu}{128}\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\widetilde{u}_3\right\|^2_{L^2}+C\left\|\partial_3 u_1^{(2)}\right\|^{\frac{5}{2}}_{L^2}\|\widetilde{u}_1\|_{L^2}\|\widetilde{u}_3\|_{L^2}.\end{aligned}$$

Due to the symmetry, the estimates of J_2 and J_3 are similar to those for J_1 and we omit further details. Collecting the bounds for J_1, J_2 and J_3, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\widetilde{u}\|^2_{L^2}+\nu\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}}\right)\widetilde{u}_1\right\|^2_{L^2}+\nu\left\|\left(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\widetilde{u}_2\right\|^2_{L^2}+\nu\left\|\left(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}}\right)\widetilde{u}_3\right\|^2_{L^2}\\ &\le C\left\|\nabla u^{(2)}\right\|^{\frac{5}{2}}_{L^2}\|\widetilde{u}\|^2_{L^2}+C\,\|\widetilde{u}_1\|^2_{L^2}\left\|\Lambda_1^{\frac{5}{4}}u_1^{(2)}\right\|^{\frac{5}{7}}_{L^2}\left\|\Lambda_1^{\frac{5}{4}}\partial_3 u_1^{(2)}\right\|^{\frac{5}{7}}_{L^2}\\ &\quad+C\,\|\widetilde{u}_2\|^2_{L^2}\left\|\Lambda_2^{\frac{5}{4}}u_2^{(2)}\right\|^{\frac{5}{7}}_{L^2}\left\|\Lambda_2^{\frac{5}{4}}\partial_1 u_2^{(2)}\right\|^{\frac{5}{7}}_{L^2}+C\,\|\widetilde{u}_3\|^2_{L^2}\left\|\Lambda_3^{\frac{5}{4}}u_3^{(2)}\right\|^{\frac{5}{7}}_{L^2}\left\|\Lambda_3^{\frac{5}{4}}\partial_2 u_3^{(2)}\right\|^{\frac{5}{7}}_{L^2}.\end{aligned}$$

Gronwall's inequality then implies that $u^{(1)}=u^{(2)}$ if $u_0^{(1)}=u_0^{(2)}$. This completes the proof of Proposition 3.1. □

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