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The Kawahara equation in weighted Sobolev spaces

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Abstract
The initial- and boundary-value problem for the Kawahara equation, a fifth-order KdV type equation, is studied in weighted Sobolev spaces. This functional framework is based on the dual-Petrov–Galerkin algorithm, a numerical method proposed by Shen (2003 *SIAM J. Numer. Anal.* **41** 1595–619) to solve third and higher odd-order partial differential equations. The theory presented here includes the existence and uniqueness of a local mild solution and of a global strong solution in these weighted spaces. If the L^2-norm of the initial data is sufficiently small, these solutions decay exponentially in time. Numerical computations are performed to complement the theory.

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1. Introduction

Fifth-order Korteweg–de Vries type equations

$$u_t - u_{xxxxx} = F(x, t, u, u_x, u_{xx}, u_{xxx})$$

arise naturally in modelling many wave phenomena (see, e.g., [5, 6, 8]). In particular, the Kawahara equation

$$u_t + uu_x + \beta u_{xxx} - u_{xxxxx} = f \tag{1.1}$$

has been derived to model magneto-acoustic waves in plasmas [6] and shallow water waves with surface tension [4]. In this equation, β is related to the Bond number in the presence of surface tension and $\beta = 0$ corresponds to the critical Bond number $\frac{1}{3}$ (see, e.g., [13]).

Our attention is focused on the initial- and boundary-value problem (IBVP) of (1.1) in the spatial domain $I = (-1, 1)$ with the boundary and initial conditions

$$\begin{cases} u(-1,t) = g(t), \quad u_x(-1,t) = h(t), \quad u(1,t) = u_x(1,t) = u_{xx}(1,t) = 0, \ \ t \geqslant 0, \\ u(x,0) = u_0(x), \quad x \in I. \end{cases} \tag{1.2}$$

Since (1.1) and (1.2) can be reformulated as an equivalent problem with homogeneous boundary conditions, we assume in the rest of the paper $g(t) = h(t) \equiv 0$. In fact, through the transform

$$u(x,t) = v(x,t) + B(x,t)$$

with

$$B(x,t) = -\frac{(x-1)^3}{4}(2g(t)+h(t)) - \frac{(x-1)^4}{16}(3g(t)+2h(t)),$$

(1.1) and (1.2) can be converted into an IBVP with homogeneous boundary condition

$$\begin{cases} v_t + B_x v + (v+B)v_x + \beta v_{xxx} - v_{xxxxx} = -B_t - BB_x - \beta B_{xxx}, \\ v(-1,t) = v_x(-1,t) = v(1,t) = v_x(1,t) = v_{xx}(1,t) = 0. \end{cases}$$

In principle, this homogeneous problem can be studied in a similar fashion as (1.1) and (1.2). There are many good reasons for studying the IBVPs rather than pure initial-value problems. For example, waves generated by a wavemaker are naturally set in a semi-infinite interval and (1.1) and (1.2) serve as a good approximate model before the waves reach the right boundary. In fact, the Kawahara equation has been studied in many works ([1, 2, 4, 5, 7, 11, 14]).

It is difficult to compute solutions of the fifth-order KdV equations numerically due to the fifth-order term. In [12] Shen proposed the dual-Petrov–Galerkin algorithm for third and higher odd-order differential equations that involves an innovative choice of test and trial functions, which allow free integration by parts without generating boundary terms. This algorithm is equivalent to the spectral-Galerkin approximation in weighted spaces. Numerical experiments involving the usual third-order KdV equation in [12] indicate that the dual-Petrov–Galerkin algorithm is very accurate and efficient. In a recent work [3], Goubet and Shen studied the IBVP for the third-order KdV equation in a functional framework based on the dual-Petrov–Galerkin method. More precisely, they established the existence and uniqueness of solutions to this IBVP in weighted Sobolev spaces.

The dual-Petrov–Galerkin algorithm was recently further developed and implemented for a fifth-order KdV equation in [15]. The goal of this paper is to build a corresponding theory on the existence and uniqueness of solutions to the IBVP (1.1) and (1.2) in Sobolev spaces. We follow the approach of Goubet and Shen [3], but the situation here is more complex. The dispersive part consists of two terms $\beta u_{xxx} - u_{xxxxx}$ and its corresponding weak formulation fails to be coercive for $\beta \leqslant -\frac{3}{80}$ (see section 2 for more details). For $\beta > -\frac{3}{80}$, the IBVP (1.1) and (1.2) is shown to possess a unique global solution for u_0 in any one of a sequence of weighted Sobolev spaces with increasing regularity (theorem 3.2). If, in addition, the L^2-norm of u_0 is small, then the solution in these weighted Sobolev spaces decays exponentially in time.

The IBVP (1.1) and (1.2) is also studied numerically to complement our theoretical results. In fact, we computed solutions of a slightly more general problem than (1.1) and (1.2). This problem involves two parameters β_1 and β_2 and the existence and uniqueness theory applies to the case when

$$\beta_1 > -\tfrac{3}{80}\beta_2. \tag{1.3}$$

We computed the solutions of this problem corresponding to β_1 and β_2 in different ranges and plotted their standard L^2-norms and weighted L^2-norms. The graphs show that the solution corresponding to β_1 and β_2 violating (1.3) may not exist for all time and thus the IBVP (1.1) and (1.2) with β violating the condition may not be globally well posed. Comparisons are also made between the weighted Sobolev norms and the standard Sobolev norms.

The rest of the paper is divided into three sections. Section 2 focuses on a weak formulation of the stationary and the linearized equation

$$\beta u_{xxx} - u_{xxxxx} = f$$

and establishes the existence and uniqueness of solutions to this formulation with any f in a weighted L^2-space (theorem 2.2). In particular, the solution operator is shown to be the generator of a contraction semi-group. Section 3 presents the existence and uniqueness results for the full IBVP (1.1) and (1.2) (theorems 3.2 and 3.4). Section 4 contains the numerical results.

2. Weak formulation of the stationary linear equation

This section presents a weak formulation of the boundary-value problem for the stationary equation

$$\begin{cases} \beta u_{xxx} - u_{xxxxx} = f, & x \in (-1, 1), \\ u(-1) = u(1) = u_x(-1) = u_x(1) = u_{xx}(1) = 0 \end{cases} \tag{2.1}$$

and establishes a theory on the existence and uniqueness of solutions to this formulation.

We first introduce some notation. Let $I = (-1, 1)$. Let $L^p(I)$ with $p \in [1, \infty]$ denote the usual Lebesgue space and $H^k(I)$ the usual L^2-based Sobolev space. Let $H_0^k(I)$ denote the completion of $C_0^\infty(I)$ under H^k-norm. For a nonnegative weight ω, define

$$\begin{aligned} L^2_\omega &= \left\{ u \in L^1_{\mathrm{loc}}(I), \int_I u^2(x)\,\omega(x)\,\mathrm{d}x < \infty \right\}, \\ V(I) &= \left\{ u \in H_0^2(I) : u_{xx} \in L^2_{\omega'} \right\}, \\ W(I) &= \left\{ u \in V(I),\ u_{xxx} \in L^2_{\frac{\omega^2}{\omega'}} \right\}. \end{aligned} \tag{2.2}$$

For the purpose of eliminating boundary terms, we choose $\omega(x) = \frac{1+x}{1-x}$. Correspondingly, $\omega'(x) = \frac{2}{(1-x)^2}$ and $\frac{\omega^2(x)}{\omega'(x)} = \frac{(1+x)^2}{2}$. In addition, we write $H(I)$ for L^2_ω and denote the inner product in H by $(\cdot, \cdot)_H$.

Lemma 2.1. *V endowed with the norm $\|u_{xx}\|_{L^2_{\omega'}}$ and W with $\|u_{xxx}\|_{L^2_{\frac{\omega^2}{\omega'}}}$ are Hilbert spaces. The embedding relations*

$$C_0^\infty \hookrightarrow W \hookrightarrow V \hookrightarrow H$$

are dense and continuous and the following Hardy type inequalities hold:

$$\int_I \frac{u^2}{(1-x)^6}\,\mathrm{d}x \leqslant \frac{4}{25}\int_I \frac{u_x^2}{(1-x)^4}\,\mathrm{d}x, \quad \int_I \frac{u_x^2}{(1-x)^4}\,\mathrm{d}x \leqslant \frac{4}{9}\int_I \frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x \qquad \forall u \in V, \tag{2.3}$$

$$r^2 \int_I \frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x - (2r + 3qr - q^2)\int_I \frac{u_x^2}{(1-x)^4}\,\mathrm{d}x + (1 - 5q + 20r)\int_I \frac{u^2}{(1-x)^6}\,\mathrm{d}x \geqslant 0 \tag{2.4}$$

for any real number r and q and

$$\int_I \frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x \leqslant \int_I u_{xxx}^2 (1+x)^2\,\mathrm{d}x \qquad \forall u \in W. \tag{2.5}$$

The general form in (2.4) is very useful and can be tailored for special needs. For example, by letting $(r,q)=(\frac{1}{2},\frac{3}{2})$ and $(r,q)=(\frac{1}{2},1)$, we have

$$\int_I \frac{u_x^2}{(1-x)^4}\,\mathrm{d}x \leqslant \frac{1}{4}\int_I \frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x + \frac{7}{2}\int_I \frac{u^2}{(1-x)^6}\,\mathrm{d}x,$$

$$\int_I \frac{u_x^2}{(1-x)^4}\,\mathrm{d}x \leqslant \frac{1}{6}\int_I \frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x + 4\int_I \frac{u^2}{(1-x)^6}\,\mathrm{d}x,$$

respectively. The proof of this lemma follows the ideas in [3] and [12]. It will be provided in the appendix for the reader's convenience.

For $u \in V$, $v \in W$ and $f \in H$, we define

$$a(u,v) = \int_I u_{xx}(-\beta(v\omega)_x + (v\omega)_{xxx})\,\mathrm{d}x \tag{2.6}$$

and consider the following weak formulation of (2.1):

$$a(u,v) = (f,v)_H.$$

We now establish the existence and uniqueness of solutions to this formulation.

Theorem 2.2. *For any $\beta > -\frac{3}{80}$ and for any $f \in H$, there exists a unique solution $u \in W$ such that*

$$a(u,v) = (f,v)_H \qquad \forall v \in W. \tag{2.7}$$

As a consequence, we can define an operator $A\colon\ D(A) \to H$ by

$$Au = f,$$

where $D(A) = \{u \in W,\ \ Au \in H\}$.

The proof of this theorem relies on the following general version of the Lax–Milgram theorem (see, e.g., [9]).

Lemma 2.3. *Let $W \subset V$ be two Hilbert spaces with W being dense and continuously embedded in V. Let $a(u,v)$ be a bilinear form on $V \times W$ satisfying*

$$a(u,v) \leqslant M\|u\|_V\|v\|_W \qquad \forall u \in V, \quad v \in W, \tag{2.8}$$

$$a(v,v) \geqslant m\|v\|_V^2 \qquad \forall v \in W, \tag{2.9}$$

where $M > m > 0$ are two constants. Then, for any $f \in V'$, there exists $u \in V$ such that

$$a(u,v) = (f,v) \qquad \forall v \in W.$$

If u is also known to be in W, then u is unique.

Proof of theorem 2.2.. It suffices to show that $a(u,v)$ defined in (2.6) verifies the condition of lemma 2.3. This can be checked directly. For $u \in V$ and $v \in W$, we can write

$$a(u,v) = \int_I u_{xx}(-\beta v_x\omega - \beta v\omega' + v_{xxx}\omega + 3v_{xx}\omega' + 3v_x\omega'' + v\omega''')\,\mathrm{d}x$$

with $\omega'' = \frac{4}{(1-x)^3}$ and $\omega''' = \frac{12}{(1-x)^4}$. The terms on the right can be bounded as follows:

$$\begin{aligned} -\beta\int_I u_{xx}v_x\omega\,\mathrm{d}x &\leqslant \frac{16}{27}\sqrt{2}\,|\beta|\left(\int_I \frac{2u_{xx}^2}{(1-x)^2}\,\mathrm{d}x\right)^{1/2}\left(\int_I \frac{v_x^2}{(1-x)^4}\,\mathrm{d}x\right)^{1/2} \\ &\leqslant \frac{32}{81}|\beta|\|u\|_V\|v\|_V, \end{aligned}$$

$$-\beta \int_I u_{xx} v\omega' \, \mathrm{d}x \leqslant 4\sqrt{2}\,|\beta| \left(\int_I \frac{2u_{xx}^2}{(1-x)^2}\,\mathrm{d}x\right)^{1/2} \left(\int_I \frac{v^2}{(1-x)^6}\,\mathrm{d}x\right)^{1/2}$$
$$\leqslant \frac{16}{15}|\beta|\|u\|_V\|v\|_V,$$

$$\int_I u_{xx} v_{xxx}\omega \, \mathrm{d}x \leqslant \left(\int_I u_{xx}^2\omega' \,\mathrm{d}x\right)^{1/2} \left(\int_I v_{xxx}^2 \frac{\omega^2}{\omega'}\,\mathrm{d}x\right)^{1/2}$$
$$= \|u\|_V\|v\|_W,$$

$$\int_I u_{xx} v_{xx}\omega' \, \mathrm{d}x \leqslant \|u\|_V\|v\|_V,$$

$$\int_I u_{xx} v_x\omega'' \, \mathrm{d}x \leqslant \left(\int_I \frac{2u_{xx}^2}{(1-x)^2}\,\mathrm{d}x\right)^{1/2} \left(\int_I \frac{8v_x^2}{(1-x)^4}\,\mathrm{d}x\right)^{1/2}$$
$$\leqslant \frac{4}{3}\|u\|_V\|v\|_V,$$

$$\int_I u_{xx} v w''' \, \mathrm{d}x \leqslant \left(\int_I \frac{2u_{xx}^2}{(1-x)^2}\,\mathrm{d}x\right)^{1/2} \left(\int_I \frac{72v^2}{(1-x)^6}\,\mathrm{d}x\right)^{1/2}$$
$$\leqslant \frac{8}{5}\|u\|_V\|v\|_V.$$

Here we have applied Hardy inequalities of lemma 2.1. According to (2.5),

$$\|v\|_V \leqslant 2\|v\|_W \qquad \forall v \in W$$

and we have thus verified (2.8) with $M = \frac{1184}{405}|\beta| + \frac{91}{5}$.

To prove (2.9), we let $v \in W$ and integrate by parts to obtain

$$\begin{aligned} a(v,v) &= \int_I v_{xx}(-\beta v_x\omega - \beta v\omega' + v_{xxx}\omega + 3v_{xx}\omega' + 3v_x\omega'' + v\omega''')\,\mathrm{d}x \\ &= \frac{3}{2}\beta\int_I v_x^2\omega'\,\mathrm{d}x - \frac{1}{2}\beta\int_I v^2 w'''\,\mathrm{d}x \\ &\quad + \frac{5}{2}\int_I v_{xx}^2\omega'\,\mathrm{d}x - \frac{5}{2}\int_I v_x^2\omega'''\,\mathrm{d}x + \frac{1}{2}\int_I v^2\omega^{(5)}\,\mathrm{d}x. \end{aligned} \tag{2.10}$$

For $\beta \geqslant 0$, we apply (2.3) to obtain

$$\frac{3}{2}\beta\int_I v_x^2\omega'\,\mathrm{d}x - \frac{1}{2}\beta\int_I v^2 w'''\,\mathrm{d}x \geqslant 3\beta\int_I \frac{v_x^2}{(1-x)^2}\,\mathrm{d}x - \frac{8}{3}\beta\int_I \frac{v_x^2}{(1-x)^2}\,\mathrm{d}x$$
$$= \frac{1}{3}\beta\int_I \frac{v_x^2}{(1-x)^2}\,\mathrm{d}x.$$

Since $\omega''' = \frac{12}{(1-x)^4}$ and $\omega^{(5)} = \frac{240}{(1-x)^6}$, we have, for $\beta \geqslant 0$,

$$a(v,v) \geqslant \frac{1}{3}\beta\int_I \frac{v_x^2}{(1-x)^2}\,\mathrm{d}x + 5\int_I \frac{v_{xx}^2}{(1-x)^2}\,\mathrm{d}x - 30\int_I \frac{v_x^2}{(1-x)^4}\,\mathrm{d}x + 120\int_I \frac{v^2}{(1-x)^6}\,\mathrm{d}x.$$

After ignoring the first term and applying (2.4) with $r = 0.4$ and $q = 1$, we get

$$a(v,v) \geqslant 0.2\int_I \frac{v_{xx}^2}{(1-x)^2}\,\mathrm{d}x = 0.1\|v\|_V^2.$$

In the case when $\beta < 0$, we apply (2.3) to obtain

$$\frac{3}{2}\beta\int_I v_x^2\omega'\,\mathrm{d}x - \frac{1}{2}\beta\int_I v^2 w'''\,\mathrm{d}x \geqslant 3\beta\int_I \frac{v_x^2}{(1-x)^2}\,\mathrm{d}x$$
$$\geqslant 12\beta\int_I \frac{v_x^2}{(1-x)^4}\,\mathrm{d}x \geqslant \frac{16}{3}\beta\int_I \frac{v_{xx}^2}{(1-x)^2}\,\mathrm{d}x.$$

Thus, for $\beta < 0$,

$$a(v,v) \geqslant \left(5+\frac{16}{3}\beta\right)\int_I \frac{v_{xx}^2}{(1-x)^2}\,\mathrm{d}x - 30\int_I \frac{v_x^2}{(1-x)^4}\,\mathrm{d}x + 120\int_I \frac{v^2}{(1-x)^6}\,\mathrm{d}x.$$

Applying (2.4) with r and q satisfying

$$1-5q+20r = 4(2r+3qr-q^2) > 0,$$

we have

$$a(v,v) \geqslant \left(5+\frac{16}{3}\beta - \frac{30r^2}{2r+3qr-q^2}\right)\int_I \frac{v_{xx}^2}{(1-x)^2}\,\mathrm{d}x.$$

In order for a to be coercive, β has to satisfy

$$\beta > \frac{15}{16}\left(\frac{6r^2}{2r+3qr-q^2} - 1\right) = \frac{15(16q^2-18q+5)}{32(5q-2)}.$$

The optimal range $\beta > -\frac{3}{80}$ is reached when $r = \frac{1}{10}$ and $q = \frac{11}{20}$, and

$$a(v,v) \geqslant \left(\frac{1}{5}+\frac{16}{3}\beta\right)\int_I \frac{v_{xx}^2}{(1-x)^2}\,\mathrm{d}x = \gamma\|v\|_V^2,$$

where

$$\gamma = \tfrac{1}{10} + \tfrac{8}{3}\beta. \tag{2.11}$$

Lemma 2.3 then implies the existence of $u \in V$ satisfying (2.7).

We now establish the uniqueness of u. If there are $u_1 \in V$ and $u_2 \in V$ satisfying (2.7), then

$$a(u_1-u_2, v) = 0 \qquad \text{for all } v \in W. \tag{2.12}$$

According to lemma 2.1, C_0^∞ is densely embedded in V; there is a sequence $v_n \in C_0^\infty$ such that

$$v_n \to u_1 - u_2 \qquad \text{in } V.$$

Since $v_n \in C_0^\infty \subset W$, we have, thanks to (2.12),

$$\gamma\|v_n\|_V^2 \leqslant a(v_n,v_n) = a(u_1-u_2,v_n) + a(v_n-(u_1-u_2),v_n)$$
$$\leqslant M\|v_n-(u_1-u_2)\|_V\|v_n\|_W. \tag{2.13}$$

Furthermore, for any $v \in W$,

$$a(v_n,v) = a(v_n-(u_1-u_2),v) \leqslant M\|v_n-(u_1-u_2)\|_V\|v\|_W \to 0 \qquad \text{as } n\to\infty.$$

In particular, for any $v \in C_0^\infty$,

$$a(v_n,v) = \int_I (v_n)_{xx}\Big(-\beta(v\omega)_x + (v\omega)_{xxx}\Big)\,\mathrm{d}x$$
$$= \int_I \Big(\beta(v_n)_{xxx} - (v_n)_{xxxxx}\Big)\,v\omega\,\mathrm{d}x \to 0 \qquad \text{as } n\to\infty. \tag{2.14}$$

Letting

$$\beta(v_n)_{xxx} - (v_n)_{xxxxx} = f_n \tag{2.15}$$

and choosing $v = f_n$ in (2.14), we have

$$\|f_n\|_H^2 = \int_I f_n^2\,\omega\,\mathrm{d}x \to 0 \qquad \text{as } n \to \infty. \tag{2.16}$$

Equations (2.15) and (2.16) allow us to show that

$$\|v_n\|_W^2 \leqslant C\|v_n\|_V(\|v_n\|_V + \|f_n\|_H). \tag{2.17}$$

Combining (2.13) and(2.17) yields

$$\gamma\|v_n\|_V^2 \leqslant C\|v_n - (u_1 - u_2)\|_V\,\|v_n\|_V^{\frac{1}{2}}\,(\|v_n\|_V + \|f_n\|_H)^{\frac{1}{2}}\,.$$

Letting $n \to \infty$, we have $v_n \to 0$ in V and consequently

$$\|u_1 - u_2\|_V \leqslant \|v_n\|_V + \|v_n - (u_1 - u_2)\|_V \to 0 \qquad \text{as } n \to \infty.$$

That is, $u_1 = u_2$ in V.

To obtain (2.17), we have used the fact that if u is smooth and satisfies

$$\beta u_{xxx} - u_{xxxxx} = f$$

with the homogeneous boundary condition, then

$$\|u\|_W^2 \leqslant C\|u\|_V(\|u\|_V + \|f\|_H). \tag{2.18}$$

We now prove this fact. Noting that $\frac{\omega^2}{\omega'} = \frac{(1+x)^2}{2}$ and integrating by parts, we have

$$\|u\|_W^2 = \int_I u_{xxx}^2\frac{\omega^2}{\omega'}\,\mathrm{d}x = -\frac{1}{2}\int_I u_{xx}u_{xxxx}(1+x)^2\,\mathrm{d}x + \frac{1}{2}\int_I u_{xx}^2\,\mathrm{d}x. \tag{2.19}$$

Obviously,

$$\frac{1}{2}\int_I u_{xx}^2\,\mathrm{d}x \leqslant \int_I \frac{2u_{xx}^2}{(1-x)^2}\,\mathrm{d}x = \|u\|_V^2. \tag{2.20}$$

By Hölder's inequality, the first term is bounded by

$$\frac{1}{2}\int_I u_{xx}u_{xxxx}(1+x)^2\,\mathrm{d}x \leqslant \|u\|_V\left(\frac{1}{8}\int_I u_{xxxx}^2(1+x)^4(1-x)^2\,\mathrm{d}x\right)^{1/2}.$$

Applying the inequality in lemma 2.4, we find

$$\frac{1}{8}\int_I u_{xxxx}^2(1+x)^4(1-x)^2\,\mathrm{d}x \leqslant \frac{1}{2}\int_I u_{xxxx}^2(1+x)^4\,\mathrm{d}x \leqslant \frac{1}{8}\int_I u_{xxxxx}^2(1+x)^6\,\mathrm{d}x.$$

Since $u_{xxxxx} = \beta u_{xxx} - f$ and

$$\begin{aligned}\frac{1}{2}\int_I(\beta u_{xxx} - f)^2(1+x)^6\,\mathrm{d}x &\leqslant \beta^2\int_I u_{xxx}^2(1+x)^6\,\mathrm{d}x + \int_I f^2(1+x)^6\,\mathrm{d}x\\ &\leqslant 16\beta^2\int_I u_{xxx}^2(1+x)^2\,\mathrm{d}x + 16\int_I f^2\frac{1+x}{1-x}\,\mathrm{d}x,\end{aligned}$$

we find

$$\begin{aligned}\frac{1}{2}\int_I u_{xx}u_{xxxx}(1+x)^2\,\mathrm{d}x &\leqslant 4\sqrt{2}|\beta|\|u\|_V\|u\|_W + 2\|u\|_V\|f\|_H\\ &\leqslant \tfrac{1}{2}\|u\|_W^2 + C\|u\|_V^2 + 2\|u\|_V\|f\|_H.\end{aligned} \tag{2.21}$$

Putting together (2.19), (2.20) and (2.21), we conclude (2.18).

This uniqueness of u allows us to show that $u \in W$. Since C_0^∞ is dense in H, we assume without loss of generality that $f \in C_0^\infty$. Because of the uniqueness, the corresponding solution u is smooth and satisfies

$$\beta u_{xxx} - u_{xxxxx} = f$$

with the homogeneous boundary condition. Therefore, $u \in W$ by (2.18). This completes the proof of theorem 2.2.

The following lemma is used in the proof of theorem 2.2.

Lemma 2.4. *If* $\|u_{xxxxx}(1+x)^3\|_{L^2(I)} < \infty$, *then*

$$\int_I u_{xxxx}^2 (1+x)^4 \,\mathrm{d}x \leqslant \frac{1}{4} \int_I u_{xxxxx}^2 (1+x)^6 \,\mathrm{d}x. \tag{2.22}$$

Proof of lemma 2.4. For u satisfying $\|u_{xxxxx}(1+x)^3\|_{L^2(I)} < \infty$, we consider

$$\begin{aligned} 0 &\leqslant \int_I \left(u_{xxxx}(1+x)^2 + u_{xxxxx}(1+x)^3\right)^2 \,\mathrm{d}x \\ &= \int_I u_{xxxx}^2 (1+x)^4 \,\mathrm{d}x + \int_I u_{xxxxx}^2 (1+x)^6 \,\mathrm{d}x + 2u_{xxxx}u_{xxxxx}(1+x)^5 \,\mathrm{d}x \\ &= -4 \int_I u_{xxxx}^2 (1+x)^4 \,\mathrm{d}x + \int_I u_{xxxxx}^2 (1+x)^6 \,\mathrm{d}x, \end{aligned} \tag{2.23}$$

which implies (2.22).

3. The full initial- and boundary-value problem

This section focuses on the full IBVP

$$\begin{aligned} &u_t + uu_x + \beta u_{xxx} - u_{xxxxx} = 0, \qquad x \in (-1, 1), \quad t > 0, \\ &u(\pm 1, t) = u_x(\pm 1, t) = u_{xx}(1, t) = 0, \qquad t > 0, \\ &u(x, 0) = u_0(x), \qquad x \in (-1, 1). \end{aligned} \tag{3.1}$$

We study its solutions at two regularity levels: mild solutions and strong solutions. For this purpose, we first examine the operator A defined in theorem 2.2. We show that $-A$ is an infinitesimal generator of a semi-group.

Theorem 3.1. *Let H, A and $D(A)$ be defined as in the previous section. Then $-A$ is an infinitesimal generator of a contraction semi-group* e^{-At}.

Proof. We apply the Hille–Yosida theorem (see, e.g., [10]). It suffices to show that A is closed, $D(A)$ is dense in H and $\|(\lambda - A)^{-1}\|_H \leqslant \frac{1}{\lambda}\|f\|_H$ for any $\lambda > 0$. That A is closed can be established by showing that A^{-1} is one to one. $D(A)$ is dense in H since $C_0^\infty \subset D(A)$. For $f \in H$, let $u = (\lambda + A)^{-1} f$. Then $(\lambda + A)u = f$ and

$$(f, u)_H = ((\lambda + A)u, u)_H = \lambda \|u\|_H^2 + (Au, u)_H.$$

Since $(Au, u)_H = a(u, u) \geqslant 0$, we obtain that $\|u\|_H \leqslant \frac{1}{\lambda}\|f\|_H$. This concludes the proof.

To study the mild solution, we first define the bilinear form

$$B(u, v) = (uv)_x, \qquad (u, v) \in V \times V.$$

Let $T > 0$. A mild solution of the IBVP (3.1) is a function $u \in C([0, T]; H) \cap L^2(0, T; V)$ satisfying

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = -B(u, u) \qquad \text{in } V', \tag{3.2}$$

$$u(0) = u_0. \tag{3.3}$$

Since $-A$ is the generator of the semi-group e^{-At}, (3.2) and (3.3) can be written in the integral form:

$$u(t) = \mathrm{e}^{-At}u_0 - \int_0^t \mathrm{e}^{-A(t-s)}B(u, u)(s)\,\mathrm{d}s.$$

We now show that the IBVP (3.1) has a unique local (in time) mild solution for any initial data $u_0 \in H$.

Theorem 3.2. *Let $\beta > -\frac{3}{80}$ and let $u_0 \in H$. Then there exists $T = T(\|u_0\|_H)$ such that the IBVP (3.1) has a unique mild solution u satisfying*

$$u \in C([0, T]; H) \cap L^2(0, T; V).$$

In addition, u obeys the bound

$$\|u(t)\|_H^2 + \gamma \int_0^t \|u(\tau)\|_V^2\,\mathrm{d}\tau \leqslant \|u_0\|_H^2 + \int_0^t \|u(\tau)\|_H^3 \|u(\tau)\|_V\,\mathrm{d}\tau, \tag{3.4}$$

where γ is as defined in (2.11).

Proof. We apply the contraction mapping principle to the integral equation

$$u(t) = \mathrm{e}^{-At}u_0 - \int_0^t \mathrm{e}^{-A(t-s)}B(u, u)(s)\,\mathrm{d}s. \tag{3.5}$$

To this end, let $X = C([0, T]; H) \cap L^2(0, T; V)$ and define, for $u \in X$,

$$\|u\|_X = \sup_{t\in[0,T]} \|u(t)\|_H + \|u\|_{L^2(0,T;V)}.$$

Let $R = \|u_0\|_H$ and $B_{2R} = \{u \in X,\ \|u\|_X \leqslant 2R\}$. We show that the right-hand side of (3.5), denoted by $G(u)$, defines a contraction mapping from B_{2R} to B_{2R}.

Let $u \in B_{2R}$. $G(u)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}G(u) + AG(u) = -B(u, u),$$

and we obtain after taking the inner product of this equation with $G(u)$ in H

$$\frac{\mathrm{d}}{\mathrm{d}t}\|G(u)\|_H^2 + 2a(G(u), G(u)) = -2(B(u, u), G(u))_H.$$

According to the proof of theorem 2.2 and the bilinear estimate in lemma 3.3,

$$2a(G(u), G(u)) \geqslant 2\gamma\|G(u)\|_V^2,$$

$$2|(B(u, u), G(u))_H| \leqslant 2\|B(u, u)\|_{V'}\|G(u)\|_V \leqslant \gamma\|G(u)\|_V^2 + C\|u\|_H^3\|u\|_V.$$

Therefore,

$$\|G(u)\|_H^2 + \gamma \int_0^t \|u(\tau)\|_V^2\,\mathrm{d}\tau \leqslant \|u_0\|_H^2 + C \int_0^t \|u(\tau)\|_H^3\|u(\tau)\|_V\,\mathrm{d}\tau.$$

If we choose $T > 0$ such that $R^2 + 16C\sqrt{T}R^4 < 2\min(1, \gamma)R^2$, then

$$\|G(u)\|_X < 2R.$$

To show G is a contraction, we first note that

$$G(u) - G(v) = -\int_0^t \mathrm{e}^{-A(t-s)}(B(u-v,u) + B(v,u-v))\,\mathrm{d}s.$$

A similar process as in the estimate of $\|G(u)\|_H$ yields

$$\begin{aligned}
\|G(u) - G(v)\|_H^2 + \gamma \int_0^t \|G(u) - G(v)\|_V^2\,\mathrm{d}s \\
&\leqslant \int_0^t (\|u-v\|_H^2\|u\|_H\|u\|_V + \|v\|_H^2\|u-v\|_H\|u-v\|_V)\,\mathrm{d}s \\
&\leqslant C\sqrt{T}\|u-v\|_X^2(\|u\|_X^2 + \|v\|_X^2).
\end{aligned}$$

If we further restrict T to $4C\sqrt{T}R^2 < \min(1,\gamma)$, then

$$\|G(u) - G(v)\|_X \leqslant \nu\|u-v\|_X,$$

where $\nu^2 = (4C\sqrt{T}R^2)/\min(1,\gamma) < 1$. Applying the contraction mapping principle completes the proof of this theorem.

In the proof of the previous theorem, we have used the following bilinear estimate.

Lemma 3.3. *For any* $(u,v) \in V \times V$,

$$\|B(u,v)\|_{V'} \leqslant C\|u\|_H\|v\|_H^{1/2}\|v\|_V^{1/2},$$

where C is a constant independent of u and v.

Proof. Let $\psi \in V$. We obtain by integrating by parts

$$(B(u,v),\psi)_H = \int_I (uv)_x\psi\omega\ \mathrm{d}x = -\int_I uv\psi_x\omega\,\mathrm{d}x - \int_I uv\psi\omega'\,\mathrm{d}x. \qquad (3.6)$$

By the first inequality in lemma 2.1,

$$\begin{aligned}
\int_I u(x)v(x)\psi_x(x)\omega(x)\,\mathrm{d}x &= \int_I u(x)\omega^{1/2}\psi_x(x)\omega'(x)v(x)\frac{\omega^{1/2}}{\omega'}\,\mathrm{d}x \\
&\leqslant \|u\|_H\|\psi_x(1-x)^{-2}\|_{L^2}\sup_{x\in I}\left|v(x)\frac{\omega^{1/2}}{\omega'}\right| \\
&\leqslant C\|u\|_H\|\psi\|_V\sup_{x\in I}\left|v(x)\frac{\omega^{1/2}}{\omega'}\right|.
\end{aligned}$$

To complete the estimate, we write

$$\begin{aligned}
\left|v(x)\frac{\omega^{1/2}}{\omega'}\right|^2 &= \frac{1}{4}v^2(x)(1+x)(1-x)^3 \\
&= \frac{1}{2}\int_{-1}^x v(y)v_y(y)(1+y)(1-y)^3\,\mathrm{d}y - \frac{1}{4}\int_{-1}^x v^2(y)(1-y)^2(2+4y)\,\mathrm{d}y.
\end{aligned}$$

It is clear that these integrals are bounded by $C(\|v\|_H\|v\|_V + \|v\|_H^2)$. Therefore,

$$\int_I u(x)v(x)\psi_x(x)\omega(x)\,\mathrm{d}x \leqslant C\|u\|_H\|\psi\|_V\|v\|_H^{1/2}\|v\|_V^{1/2}.$$

The second term in (3.6) can be bounded similarly. In fact,

$$\int_I uv\psi\omega'\,\mathrm{d}x \leqslant C\|u\|_H\|v\|_H\sup_{x\in I}|\psi(x)\omega'(x)\omega^{-1}(x)|.$$

To show that $\sup_{x\in I}|\psi(x)\omega'(x)\omega^{-1}(x)| \leqslant C$, we note that $\omega'(x)\omega^{-1}(x) = 2(1-x)^{-1}(1+x)^{-1}$ and show that

$$\frac{\psi^2(x)}{(1-x)^4} \in W^{1,1}(I) \quad \text{and} \quad \frac{\psi^2(x)}{(1+x)^4} \in W^{1,1}(I).$$

Then the embedding $W^{1,1}(I) \subset L^\infty(\bar{I})$ leads to the conclusion. By the first inequality in lemma 2.1,

$$\int_I \frac{\psi^2(x)}{(1-x)^4}\,\mathrm{d}x \leqslant 4\int_I \frac{\psi^2(x)}{(1-x)^6}\,\mathrm{d}x \leqslant \frac{32}{225}\|\psi\|_V^2.$$

In addition,

$$\partial_x\left(\frac{\psi^2(x)}{(1-x)^4}\right) = \frac{2\psi(x)\psi_x(x)}{(1-x)^4} + \frac{4\psi^2(x)}{(1-x)^5}$$

and

$$\int_I \frac{2\psi(x)\psi_x(x)}{(1-x)^4}\,\mathrm{d}x \leqslant C\|\psi(1-x)^{-3}\|_{L^2}\|\psi_x(1-x)^{-2}\|_{L^2} \leqslant C\|\psi\|_V^2,$$

$$\int_I \frac{4\psi^2(x)}{(1-x)^5}\,\mathrm{d}x \leqslant C\|\psi(1-x)^{-3}\|_{L^2}^2 \leqslant C\|\psi\|_V^2.$$

Therefore, $\frac{\psi^2(x)}{(1-x)^4} \in W^{1,1}(I)$ and similarly $\frac{\psi^2(x)}{(1+x)^4} \in W^{1,1}(I)$. This concludes the proof of lemma 3.3.

We now study solutions of the IBVP (3.1) in a stronger sense and establish the global existence and uniqueness of such solutions. To this end, we define

$$H_1(I) = \{u \in H(I),\ u_x \in H(I)\} \quad \text{and} \quad V_1(I) = \{u \in V(I),\ u_x \in V(I)\}.$$

Theorem 3.4. *Assume $\beta > -\frac{3}{80}$ and $u_0 \in H_1(I) \cap L^2(I)$. Let $T > 0$ be arbitrarily fixed. Then the IBVP (3.1) has a unique solution u satisfying*

$$u \in C([0,T]; H_1 \cap L^2) \cap L^2(0,T; V_1).$$

Furthermore, if the L^2-norm of u_0 is small in the sense that

$$\|u_0\|_{L^2(I)} \leqslant C\gamma \tag{3.7}$$

for some suitable constant C, then $\|u(t)\|_H$ and $\|u_x(t)\|_H$ decay exponentially in time.

Proof. Since $u_0 \in H$, theorem 3.2 asserts the existence of a local solution u satisfying

$$u \in C([0,T]; H) \cap L^2(0,T; V). \tag{3.8}$$

Thanks to $u_0 \in L^2(I)$, u obeys the global *a priori* bound

$$\|u(t)\|_{L^2(I)} \leqslant \|u_0\|_{L^2} \qquad \text{for all} \quad t > 0. \tag{3.9}$$

This can be established by first noticing that smooth solutions of (3.2) satisfy

$$\|u(t)\|_{L^2}^2 + \int_0^t u_{xx}^2(0,\tau)\,\mathrm{d}\tau = \|u_0\|_{L^2}^2$$

and then going through a limiting process. We now apply (3.9) to show that, for $t \leqslant T$,

$$\|u(t)\|_H \leqslant C(T)\|u_0\|_H, \tag{3.10}$$

where $C(T)$ is a constant depending on T only. Taking the inner product of (3.2) with u in H and applying lemma 3.5, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_H^2 + 2\gamma\|u\|_V^2 \leqslant C\|u\|_{L^2}\|u\|_H\|u\|_V. \tag{3.11}$$

Inserting the inequality

$$C\|u\|_{L^2}\|u\|_H\|u\|_V \leqslant \gamma\|u\|_V^2 + \tfrac{1}{4}C^2\gamma^{-1}\|u\|_{L^2}^2\|u\|_H^2$$

in (3.11) and applying Gronwall's inequality, we obtain (3.10). If $\|u_0\|_L^2$ satisfies (3.7), (3.9) and (3.10) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_H^2 + (2\gamma - C\|u_0\|_{L^2})\,\|u\|_V^2 \leqslant 0,$$

where we have used $\|u\|_H \leqslant \|u\|_V$. Consequently, $\|u(t)\|_H$ decays exponentially in time.

We further show that, for some constant C depending on T only,

$$\|u_x(t)\|_H \leqslant C(T)\|u_{0x}\|_H. \tag{3.12}$$

To prove (3.12), we start with the equation that $v = u_x$ satisfies

$$\frac{\mathrm{d}v}{\mathrm{d}t} + Av = -(uv)_x.$$

Taking the inner product with v in H and applying lemma 3.5, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_H^2 + 2\gamma\|v\|_V^2 \leqslant C\|u\|_H\|v\|_V^2.$$

The desired inequality then follows from Gronwall's inequality. This concludes the proof of theorem 3.4.

The following estimates have been used in the proof of theorem 3.4.

Lemma 3.5. *The constants C in the bounds are absolute constants.*

(1) For any $u \in V$,

$$\left|\int_I u^2 u_x \omega \,\mathrm{d}x\right| \leqslant C\|u\|_{L^2}\|u\|_H\|u\|_V. \tag{3.13}$$

(2) For $u \in H$ and $v \in V$,

$$\left|\int_I (uv_x)_x v\omega \,\mathrm{d}x\right| \leqslant C\|u\|_H\|v\|_V^2. \tag{3.14}$$

The proof of this lemma will be given in the appendix.

Finally we remark that theorems 3.2 and 3.4 can be easily extended to a slightly more general problem than (3.1). In fact, the following corollaries can be established by modifying the proofs of theorems 3.2 and 3.4.

Corollary 3.6. *Let $L > 0$ and $J = (-L, L)$. Let $\beta_2 > 0$ and consider*

$$\begin{aligned} &u_t + uu_x + \beta_1 u_{xxx} - \beta_2 u_{xxxxx} = 0, \qquad x \in J,\ t > 0,\\ &u(\pm L, t) = u_x(\pm L, t) = u_{xx}(L, t) = 0, \qquad t > 0,\\ &u(x, 0) = u_0(x), \qquad x \in J. \end{aligned} \tag{3.15}$$

Assume $u_0 \in H(J)$ and

$$L^2\beta_1 > -\tfrac{3}{80}\beta_2. \tag{3.16}$$

Then there exists $T = T(\|u_0\|_{H(J)})$ such that the IBVP (3.15) has a unique mild solution u satisfying

$$u \in C([0,T]; H(J)) \cap L^2(0,T; V(J)).$$

In addition, u obeys the bound

$$\|u(t)\|_H^2 + \mu\int_0^t \|u(\tau)\|_V^2\,\mathrm{d}\tau \leqslant \|u_0\|_H^2 + \int_0^t \|u(\tau)\|_H^3 \|u(\tau)\|_V\,\mathrm{d}\tau,$$

where $\mu = \frac{5}{14}\beta_2 + \frac{8}{3}L^2\min(0,\beta_1)$.

Corollary 3.7. *Consider the IBVP (3.15). Assume β_1 and β_2 satisfying (3.16) and $u_0 \in H_1(J) \cap L^2(J)$. Let $T > 0$ be arbitrarily fixed. Then the IBVP (3.15) has a unique solution u satisfying*

$$u \in C([0,T]; H_1(J) \cap L^2(J)) \cap L^2(0,T; V_1(J)).$$

Furthermore, if the L^2-norm of u_0 is small in the sense that

$$\|u_0\|_{L^2(J)} \leqslant C\gamma$$

for some suitable constant C, then $\|u(t)\|_H$ and $\|u_x(t)\|_H$ decay exponentially in time.

4. Numerical results

This section numerically studies the behaviour of solutions of (3.15) with $L = 1$ and for β_1 and β_2 in different ranges. The numerical scheme is the dual Petrov–Galerkin algorithm that has previously been developed in [12] and [15]. The results presented here indicate clearly that solutions of (3.15) with β_1 and β_2 violating (3.16) may not exist for all time.

First, we compute the solution of the Kawahara equation

$$u_t + u\,u_x + \frac{1}{M^2}u_{xxx} - \frac{1}{M^4}u_{xxxxx} = 0, \qquad x \in (-1,1), \quad t \in [0,100] \tag{4.1}$$

with zero boundary data and with the initial data

$$u(x,0) = u_{\mathrm{ex}}(x,0), \tag{4.2}$$

where

$$u_{\mathrm{ex}}(x,t) = \frac{105}{169}\,\mathrm{sech}^4\left[\frac{M}{2\sqrt{13}}\left(x - \frac{36t}{169}\right)\right] \tag{4.3}$$

is an exact soliton solution of (4.1) before it hits the right boundary.

In (4.1), $\beta_1 = 1/M^2$ and $\beta_2 = 1/M^4$ and they trivially satisfy condition (3.16). Corollary 3.7 assesses that the IBVP (4.1) and (4.2) has a global solution. We take $M = 200$ and plot the standard norm $\|u(\cdot,t)\|_{L^2}$ versus the weighted norm $\|u(\cdot,t)\|_{L^2_\omega}$ and $\|u_x(\cdot,t)\|_{L^2}$ versus $\|u_x(\cdot,t)\|_{L^2_\omega}$ as functions of t (figures 1 and 2). We take the number of modes N in the dual-Petrov–Galerkin scheme to be 1000 and the time step $\Delta t = 0.001$.

The solution of (4.1) and (4.2) is the solitary wave given by (4.3) and its standard L^2-norm remains a constant before it hits the right boundary. Its L^2-norm starts decaying after it reaches the boundary. The weighted L^2-norm decays exponentially after the time when the wave hits the right boundary. The derivative of the solution (in both norms) also decays in time after an initial surge.

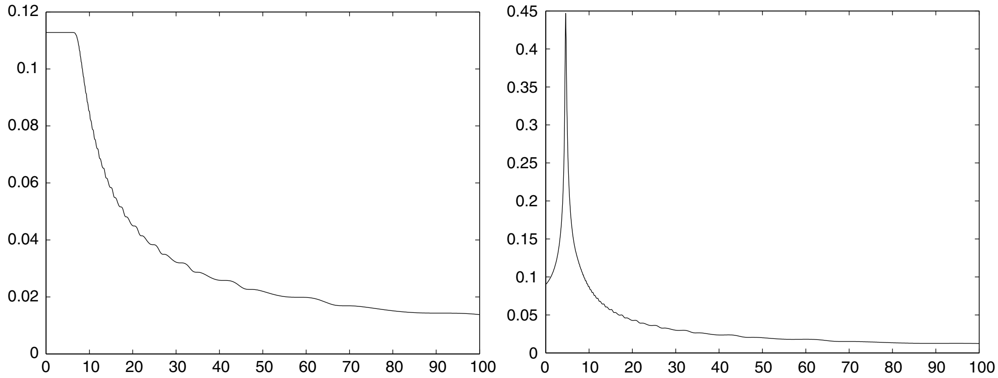

Figure 1. Left: $\|u(\cdot,t)\|_{L^2}$ versus t. Right: $\|u(\cdot,t)\|_{L^2_\omega}$ versus t.

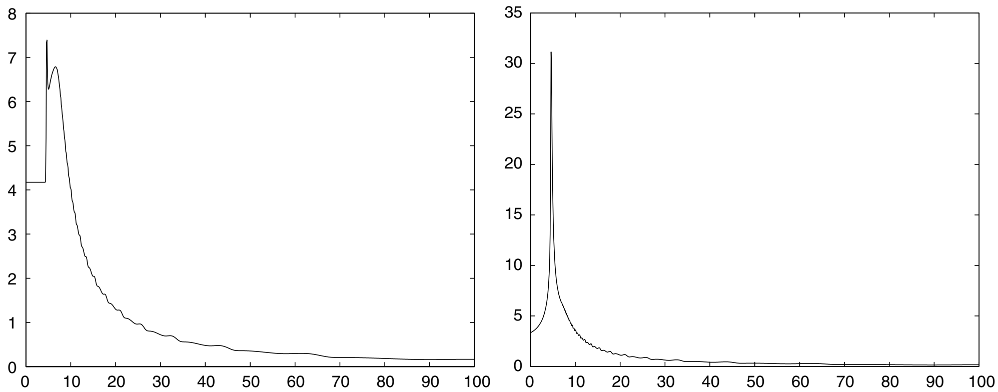

Figure 2. Left: $\|u_x(\cdot,t)\|_{L^2}$ versus t. Right: $\|u_x(\cdot,t)\|_{L^2_\omega}$ versus t.

Second, we examine the solution of the Kawahara equation:

$$\begin{aligned} & u_t + u\,u_x + \beta_1\,u_{xxx} - \beta_2\,u_{xxxxx} = 0, \qquad x \in (-1,1), \quad t \in [0,10], \\ & u(x,0) = u_{\rm ex}(x,0), \end{aligned} \tag{4.4}$$

where $\beta_1 = -0.01$ and $\beta_2 = 1/M^4$, and the boundary conditions are set to be homogeneous. It is clear that β_1 and β_2 violate (3.16) when

$$M \geqslant \left(\tfrac{15}{4}\right)^{1/4} \approx 1.3916$$

and the existence and uniqueness theory presented in the previous section does not cover this case. To see how the solution behaves, we choose $M = 200$ and plotted both the L^2-norm $\|u(\cdot,t)\|_{L^2}$ and the weighted L^2-norm $\|u(\cdot,t)\|_{L^2_\omega}$ (figure 3). The graphs clearly show that both norms quickly grow in time after an initial decay. This is an indication that the IBVP (3.15) may not be globally well posed when condition (3.16) is not met.

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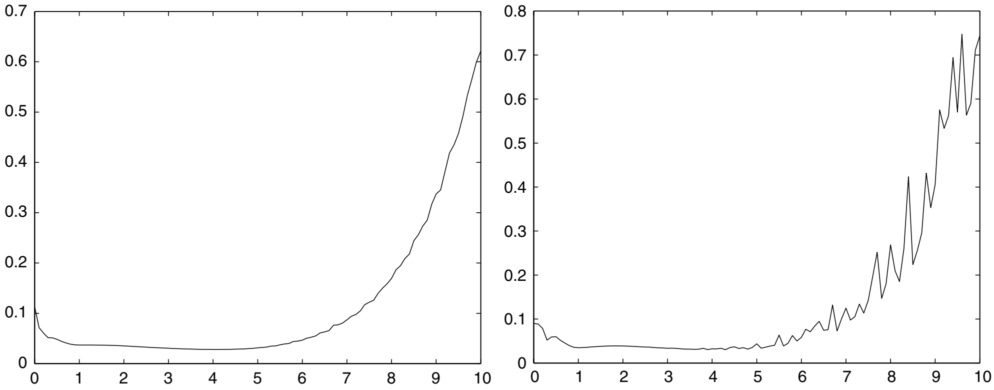

Figure 3. Left: $\|u(\cdot,t)\|_{L^2}$ versus t. Right: $\|u(\cdot,t)\|_{L^2_\omega}$ versus t.

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Appendix

We provide in this appendix the proofs of lemmas 2.1 and 3.5.

Proof of lemma 2.1. The proof follows the ideas of Goubet and Shen [3] and Shen [12]. First of all, $\|u_{xx}\|_{L^2_{w'}}$ and $\|u_{xxx}\|_{L^2_{\frac{w^2}{w'}}}$ are clearly norms in V and W, respectively. To show that C_0^∞ is dense in V, it suffices to show $C_0^\infty(I)^\perp = \{0\}$. To this end, let $u \in C_0^\infty(I)^\perp$, namely,

$$\int_I u_{xx}\phi_{xx}\omega'(x)\,\mathrm{d}x = 0 \qquad \text{for all } \phi \in C_0^\infty.$$

Since $\omega'(x) = 2/(1-x)^2$, we obtain by integrating by parts

$$\partial_{xx}\left(u_{xx}(1-x)^{-2}\right) = 0,$$

which implies

$$u_{xx} = a(1-x)^3 + b(1-x)^2.$$

Therefore, for some constants c and d,

$$u(x) = \frac{1}{20}a(1-x)^5 + \frac{b}{12}(1-x)^4 + c(1-x) + d.$$

The boundary conditions $u(\pm 1) = u_x(\pm 1) = u_{xx}(1) = 0$ imply that $a = b = c = d = 0$. That is, $u = 0$. Similar arguments show that C_0^∞ is dense in H and in W.

We now prove inequalities (2.3), (2.4) and (2.5). Since C_0^∞ are dense in W, V and H, it suffices to prove them for $u \in C_0^\infty$. To prove (2.3), we have, for any number a,

$$\begin{aligned} 0 &\leqslant \int_I \left(\frac{u}{1-x} + a u_x\right)^2 \frac{1}{(1-x)^4}\,\mathrm{d}x \\ &= \int_I \frac{u^2}{(1-x)^6}\,\mathrm{d}x + 2a\int_I \frac{u\,u_x}{(1-x)^5}\,\mathrm{d}x + a^2\int_I \frac{(u_x)^2}{(1-x)^4}\,\mathrm{d}x. \end{aligned}$$

Integrating by parts in the second term leads to

$$2a\int_I \frac{u\,u_x}{(1-x)^5}\,\mathrm{d}x = -5a\int_I \frac{u^2}{(1-x)^6}\,\mathrm{d}x.$$

Taking $a=\frac{2}{5}$ yields the first inequality in (2.3). Similarly, we can show

$$\int_I \frac{u_x^2}{(1-x)^4}\,\mathrm{d}x \leqslant \frac{4}{9}\int_I \frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x \qquad \text{for all } u\in V.$$

To prove (2.4), it suffices to consider

$$\begin{aligned}0 &\leqslant \int_I \left(\frac{u}{(1-x)^2}+\frac{qu_x}{(1-x)}+ru_{xx}\right)^2\frac{1}{(1-x)^2}\,\mathrm{d}x\\ &= (1-5q+20r)\int_I \frac{u^2}{(1-x)^6}\,\mathrm{d}x-(2r+3qr-q^2)\int_I\frac{u_x^2}{(1-x)^4}\,\mathrm{d}x + r^2\int_I \frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x.\end{aligned}$$

In particular, when $q=\frac{3}{2}$ and $r=\frac{1}{2}$, we have

$$0\leqslant \frac{7}{2}\int_I\frac{u^2}{(1-x)^6}\,\mathrm{d}x-\int_I\frac{u_x^2}{(1-x)^4}\,\mathrm{d}x+\frac{1}{4}\int_I\frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x.$$

Equation (2.5) is obtained by considering

$$\begin{aligned}0 &\leqslant \int_I\left(u_{xxx}(1+x)+\frac{u_{xx}}{1-x}\right)^2\,\mathrm{d}x\\ &= \int_I u_{xxx}^2(1+x)^2\,\mathrm{d}x + 2\int_I\frac{1+x}{1-x}u_{xx}u_{xxx}\,\mathrm{d}x+\int_I\frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x\end{aligned}$$

and integrating by parts in the second term.

To see that $V\hookrightarrow H$, we apply (2.3) to obtain

$$\|u\|_H^2 = \int_I u^2(x)\omega(x)\,\mathrm{d}x \leqslant C\int_I\frac{u^2}{(1-x)^6}\,\mathrm{d}x\leqslant C\int_I\frac{u_{xx}^2}{(1-x)^2}\,\mathrm{d}x = C\|u\|_V^2.$$

This concludes the proof of lemma 2.1.

Proof of lemma 3.5. Integrating by parts and applying Hölder's inequality, we have

$$\int_I u^2u_x\omega\,\mathrm{d}x\leqslant ||u||_{L^2}\|u\|_H\sup_{x\in I}|u_x\omega^{1/2}|. \tag{A.1}$$

To bound $\sup_{x\in I}|u_x\omega^{1/2}|$, we apply (2.3) in lemma 2.1 to obtain

$$\begin{aligned}u_x^2(x)\omega(x) &= 2\int_{-1}^x u_xu_{xx}\frac{1+x}{1-x}\,\mathrm{d}x+\int_{-1}^x\frac{2u_x^2}{(1-x)^2}\,\mathrm{d}x\\ &\leqslant 16\|u_x(1-x)^{-2}\|_{L^2}\|u_{xx}(1-x)^{-1}\|_{L^2}+8\|u_x(1-x)^{-2}\|_{L^2}\\ &\leqslant C\|u\|_V^2.\end{aligned}$$

Inserting this bound in (A.1) yields (3.13). Equation (3.14) can be established similarly. In fact,

$$\begin{aligned}\int_I(uv_x)_xv\omega\,\mathrm{d}x &= -\int_I uv_x^2\omega\,\mathrm{d}x-\int_I uv\,v_xw'\,\mathrm{d}x\\ &\leqslant C\|u\|_H\|v\|_V\sup_{x\in I}|v_x|+C\|u\|_H\|v\|_V\sup_{x\in I}|v\omega^{-1/2}|.\end{aligned} \tag{A.2}$$

$\sup_{x\in I} |v\omega^{-1/2}|$ can be bounded as in lemma 3.3,

$$\sup_{x\in I} |v\omega^{-1/2}| \leqslant C\|v\|_V.$$

$\sup_{x\in I} |v_x|$ can be estimated as follows:

$$\begin{aligned} v_x^2(x) = 2\int_{-1}^{x} v_x v_{xx}\,\mathrm{d}x &\leqslant 16\int_{-1}^{x} \frac{|v_x|}{(1-x)^2}\frac{|v_{xx}|}{(1-x)}\,\mathrm{d}x \\ &\leqslant 16\|v_x(1-x)^{-2}\|_{L^2}\,\|v_{xx}(1-x)^{-1}\|_{L^2} \\ &\leqslant \frac{16}{3}\|v\|_V^2. \end{aligned}$$

Inserting these bounds in (A.2) leads to (3.14).

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