Journal of Mathematical Physics

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Citation: J. Math. Phys. 53, 053701 (2012); doi: 10.1063/1.4711133
View online: http://dx.doi.org/10.1063/1.4711133
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v53/i5
Published by the American Institute of Physics.

JOURNAL OF MATHEMATICAL PHYSICS **53**, 053701 (2012)

The generalized Buckley-Leverett and the regularized Buckley-Leverett equations

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(Received 7 August 2011; accepted 17 April 2012; published online 7 May 2012)

This paper studies solutions of the generalized Buckley-Leverett (BL) equation with variable porosity and solutions of the regularized BL equation with the Burgers-Benjamin-Bona-Mahony-type regularization. We construct global in time weak solutions to the Cauchy problem and to the initial- and boundary-value problem for the generalized BL equation and global classical solutions for the regularized BL equation. Solutions of the regularized BL equation are shown to converge to the corresponding solution of the generalized BL equation when the coefficient γ of the BBM term and the coefficient ν of the Burgers term obey $\gamma = O(\nu^2)$. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4711133]

I. INTRODUCTION

The standard Buckley-Leverett (BL) equation

$$\partial_t u + \partial_x (f(u)) = 0$$

with

$$f(u) = \frac{u^2}{u^2 + M(1-u)^2}, \quad M > 0$$

models two-phase flow such as oil and water in a porous medium like soil and rock.[11] It expresses the conservation of mass for the saturation of water u where f denotes the water fractional flow function and M the water over oil viscosity ratio. The BL equation and some of its regularizations are not only useful in many practical applications but also important as a model of conservation laws and regularized conservation laws (see, e.g., Refs. 1, 2, 11, 15, 17, 34, and 38–40).

This paper is devoted to the generalized BL equation with variable porosity (see, e.g., Refs. 38 and 39)

$$\partial_t u + \partial_x \Big(f(\phi, u)\big(1 + \frac{k}{\mu_0}(1 - \frac{u}{\phi})^2 \partial_x P_c\big)\Big) = 0 \tag{1.1}$$

and the regularized BL equation

$$\partial_t u + \partial_x f(\phi, u) = \nu \partial_{xx} u + \gamma \partial_{xxt} u, \tag{1.2}$$

where

$$f(\phi, u) = \frac{u^2}{u^2 + M(\phi - u)^2}, \quad 0 \le u \le \phi, \quad M > 0. \tag{1.3}$$

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Here $\phi = \phi(x) > 0$ is the porosity of the medium, $u := \phi(x)\tilde{u}$ with $\tilde{u}$ being the saturation of water, k, μ_0 are constants representing the absolute permeability and the viscosity of water, respectively, P_c is the dynamic capillary pressure which depends on ϕ and u, and $\nu \geq 0$ and $\gamma \geq 0$ are real parameters with at least one of them nonzero.

Our goal is several fold. First, we would like to understand the solution of (1.1). We start with the case when $\frac{\partial P_c}{\partial x} = 0$, namely,

$$\partial_t u + \partial_x f(\phi, u) = 0. \tag{1.4}$$

We consider both the Cauchy problem

$$\begin{cases} u_t + f(\phi, u)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{1.5}$$

and the initial and boundary value problem (IBVP),

$$\begin{cases} u_t + f(\phi, u)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^+, \\ u(0,t) = u_B(t), & t \in \mathbb{R}^+, \end{cases} \tag{1.6}$$

where $u_0(0) = u_B(0)$. We establish the global existence of weak solutions of (1.5) and of (1.6) when the total variations of the data are finite. The approach is Glimm's constructive method (see, e.g., Refs. 16,18,19,21,23–27,30,31,33, and 36). However, since $f(\phi, u)$ is not a convex function of u and ϕ is not constant, suitable modifications have to be made in order for Glimm's method to apply. We then turn to the case when $\frac{\partial P_c}{\partial x}$ is a nonconstant function of x, t. The situation is even more complex since now the flux function

$$\widetilde{F}(x,t,\phi,u) = f(\phi,u)\Big(1 + \frac{k}{\mu_0}(1 - \frac{u}{\phi})^2 \frac{\partial P_c}{\partial x}\Big)$$

involves the variables x and t. Following the idea introduced by Dafermos-Hsiao,[14] we freeze the variables x and t in each grid and considering the modified Riemann problem at the center of each cell. This process still allows us to construct a sequence of approximate solutions, which enables us to extend Glimm's method to the equation with flux $\widetilde{F}(x,t,\phi,u)$. Analogous techniques can also be found in Refs. 13 and 20. More technical details are left to the construction in Sec. II. It is worth pointing out that Temple[37] has previously studied solutions of the BL equation with constant porosity. The BL equation studied in Ref. 37 was expressed as a 2×2 non-strictly hyperbolic system or Temple system

$$\begin{cases} \partial_t s + \partial_x (s\, G(s,b)) = 0, \\ \partial_t b + \partial_x (b\, G(s,b)) = 0. \end{cases} \tag{1.7}$$

Temple established the global existence of weak solutions of the Cauchy problem for (1.7) by introducing a new type of Glimm functional to achieve the stability for the non-strictly hyperbolic systems.

Second, we establish the global in time existence and uniqueness of solutions to the Cauchy problem and the IBVP for the regularized BL equation in (1.2). Due to the regularization terms on the right-hand side of (1.2), solutions of (1.2) are more regular than the counterparts of (1.1). The proof of these existence and uniqueness results involves two steps: the local existence through the contraction mapping principle and the global *a priori* bounds. We remark that the well posedness of the Cauchy problems and the IBVPs for equations with the Burgers-BBM type regularization have been studied extensively and many interesting results are available (see, e.g., Refs. 3–8 and 11). In addition, we obtain the eventual periodicity of solutions to the initial and boundary problems for (1.2). That is, any periodic boundary datum yields an eventually periodic solution. The eventual periodicity is an interesting phenomenon observed in laboratory experiments involving water waves and has been investigated in several recent papers (see Refs. 9, 10, and 22).

Our third goal is to determine if solutions of (1.2) converge to the counterpart of (1.1). This is a nontrivial issue. The pioneering work of Schonbek[35] sets up a framework for handling such convergence problems and establishes the convergence of solutions of the generalized KdV-Burgers equations and of the generalized BBM-Burgers equations to those of the corresponding conservation laws. In particular, any solution of

$$\partial_t u + \partial_x(f(u)) = \nu \partial_{xx} u + \gamma \partial_{xxt} u$$

with

$$|f(u)| \le C(1+|u|^p) \text{ for } p \ge 2 \quad \text{and} \quad \gamma = O(\nu^4)$$

converges to a weak solution of the conservation law $\partial_t u + \partial_x(f(u)) = 0$. The approach of Schonbek was further explored and applied to generalized versions of the KdV-Burgers equation by LeFloch and Natalini[29] and LeFloch and Kondo.[28] The convergence issue concerning the regularized BL equation (1.2) can still be studied through the method of Schonbek. What is special with (1.2) is that the flux function $f(\phi, u)$ is not a convex function of u but bounded uniformly in u. It is shown here that, if $\gamma = O(\nu^2)$, any sufficiently regular solution $u^{\nu,\gamma}$ of the Cauchy problem for (1.2) converges in L^p for any $1 < p < 2$ to a weak solution of (1.4). We note that the weaker assumption $\gamma = O(\nu^2)$ is due to the slower growth rate of $f(\phi, u)$ in u.

To complement our theoretical results and to directly visualize the behavior of solutions to the BL and the regularized BL equations, we have numerically computed their solutions corresponding to two representative initial data. Detailed comparisons are made to understand how the parameters ν and γ affect the solutions. In particular, the limiting behavior of solutions of the regularized BL equations is studied to check if the assumption $\gamma = O(\nu^2)$ in our convergence theory is sharp. There is an indication that $\gamma = O(\nu^2)$ may not be removed.

The rest of this paper is divided into four sections. Section II presents the global existence of weak solutions to the Cauchy problem and to the IBVP for (1.1). Section III is concerned with the Cauchy problem and the IBVP for the regularized BL equation (1.2). Global in time solutions and the eventual periodicity are established. Section IV proves the convergence of solutions of (1.2) to the corresponding ones of (1.4) for the Cauchy problem. Section V presents some numerical experiments that complement the theoretical results on the convergence issue.

II. THE GENERALIZED BL EQUATION WITH VARIABLE POROSITY

In this section, we consider the following generalized BL equation with variable porosity:[38,39]

$$\phi\, \partial_t \tilde{u} + \partial_x \Big(\tilde{f}(\tilde{u}) \big(1 + \frac{k}{\mu_0}(1-\tilde{u})^2 \frac{\partial P_c}{\partial x}\big)\Big) = 0, \tag{2.1}$$

where $\phi = \phi(x) > 0$ is the porosity of soil and $\tilde{u}$ is the saturation of water satisfying $0 < \tilde{u} \le 1$, k, μ_0 are constants representing, respectively, the absolute permeability and the viscosity of water, P_c is the dynamic capillary pressure. Flux $\tilde{f}$ is given by

$$\tilde{f}(\tilde{u}) = \frac{\tilde{u}^2}{\tilde{u}^2 + M(1-\tilde{u})^2}, \quad 0 < \tilde{u} \le 1, \tag{2.2}$$

where $M > 0$ is the ratio of viscosities between water and oil. Writing (2.1) in conservative form, we obtain the equivalent equation of (2.1),

$$\partial_t u + \partial_x \Big(f(\phi, u) \big(1 + \frac{k}{\mu_0}(1 - \frac{u}{\phi})^2 \frac{\partial P_c}{\partial x}\big)\Big) = 0, \tag{2.3}$$

where $u := \phi(x)\tilde{u}$, and

$$f(\phi, u) = \frac{u^2}{u^2 + M(\phi - u)^2}, \quad 0 < u \le \phi. \tag{2.4}$$

In general, P_c is a function of ϕ, u, and u_t, which means that (2.3) is not of hyperbolic type when $\frac{k}{\mu_0}$ is large. To demonstrate the properties of hyperbolic waves, we assume that P_c is a given C^1 function of x, t and $\frac{k}{\mu_0}\frac{\partial P_c}{\partial x}$ is sufficiently small.

Let

$$\varepsilon := \frac{k}{\mu_0} > 0, \quad h(x,t) := \frac{\partial P_c}{\partial x} \geq 0. \tag{2.5}$$

To obtain the admissible waves for Riemann and Cauchy problems, we assume that there exists some positive constant ϵ_* such that, for any $(x, t) \in \mathbb{R} \times \mathbb{R}^+$,

$$0 \leq \varepsilon h(x,t) < \varepsilon_* < M. \tag{2.6}$$

Then (2.3) can be written as

$$u_t + \tilde{F}(\varepsilon, t, x, \phi, u)_x = 0, \tag{2.7}$$

where

$$\tilde{F}(\varepsilon, t, x, \phi, u) = f(\phi, u)\big(1 + \varepsilon h(x,t)(1 - \frac{u}{\phi})^2\big). \tag{2.8}$$

We call Eq. (2.7) the zero order generalized BL equation if $h(x, t) = 0$. In such case, (2.7) is reduced to

$$u_t + f(\phi, u)_x = 0. \tag{2.9}$$

Or else, (2.7) is called the first order generalized BL equation which can be considered as the perturbation of (2.9) when $\varepsilon h(x, t)$ is sufficiently small. We divide the global existence results for generalized BL equation into Secs. II A–II C.

A. The zero order generalized BL equation and Riemann problem

In this subsection, we wish to establish the global existence of weak solutions to the Cauchy and initial-boundary value problems of (2.7) and (2.9). We first consider the Cauchy problem of (2.9),

$$\begin{cases} u_t + f(\phi, u)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.10}$$

where $\phi(x)$ is Lipschitz continuous, and $\phi(x)$ and $u_0(x)$ are of finite total variation. The Riemann problem of the generalized BL equation is a special type of (2.10) given by

$$\begin{cases} u_t + f(\phi, u)_x = 0, & x \in \mathbb{R} \times \mathbb{R}^+, \\ (\phi(x), u(x,0)) = \begin{cases} (\phi_L, u_L), & x < 0, \\ (\phi_R, u_R), & x > 0, \end{cases} \end{cases} \tag{2.11}$$

where ϕ_L, ϕ_R, u_L, and u_R are constants satisfying $0 < u_L \leq \phi_L$, $0 < u_R \leq \phi_R$. We say a L^1 function u is a weak solution of (2.10) if for any $\psi \in C_0^\infty(\mathbb{R} \times [0, \infty))$, u satisfies

$$R_\psi(\phi, u) := \iint_{t>0} u\psi_t + f(\phi, u)\psi_x dx dt + \int_{-\infty}^{\infty} u_0(x)\psi(x, 0) dx = 0. \tag{2.12}$$

We extend the Glimm method[16] to establish the global existence of weak solutions to (2.10). First, we augment the equation in (2.10) by adding $\phi_t = 0$. Then, we obtain the equivalent 2×2 system of conservation laws

$$U_t + F(U)_x = 0, \tag{2.13}$$

where $U := (\phi, u)^T$, $F(U) := (0, f(\phi, u))^T$. Therefore, problem (2.10) is equivalent to the following Cauchy problem:

$$\begin{cases} U_t + F(U)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ U(x, 0) = U_0(x) = (\phi(x), u_0(x))^T, & x \in \mathbb{R}. \end{cases} \tag{2.14}$$

The corresponding Riemann problem is

$$\begin{cases} U_t + F(U)_x = 0, & x \in \mathbb{R} \times \mathbb{R}^+, \\ U(x,0) = \begin{cases} U_L := (\phi_L, u_L)^T, & x < 0, \\ U_R := (\phi_R, u_R)^T, & x > 0. \end{cases} \end{cases} \tag{2.15}$$

The eigenvalues and corresponding eigenvectors of Jacobian matrix $DF(U)$ are

$$\lambda_0(U) = 0, \quad \lambda_1(U) = f_u(U) = \tfrac{2M\phi u(\phi-u)}{g^2(\phi,u)} \geq 0,$$
$$r_0(U) = (f_u, -f_\phi)^T, \quad r_1(U) = (0,1)^T, \tag{2.16}$$

where $g(\phi, u) := u^2 + M(\phi - u)^2$. Therefore, by direct calculation we obtain

$$\nabla\lambda_0 \cdot r_1(U) = 0,$$
$$\nabla\lambda_1 \cdot r_1(U) = f_{uu}(U) = 2M\phi \cdot \{\tfrac{2(M+1)u^3 - 3(M+1)\phi u^2 + M\phi^3}{g^3(\phi,u)}\} = 2M\phi^4 \cdot \tfrac{h(\tilde{u})}{g^3(\phi,u)}, \tag{2.17}$$

where $h(\tilde{u}) := 2(M+1)\tilde{u}^3 - 3(M+1)\tilde{u}^2 + M$. It is easy to show that there is a unique $u_* \in (0, \phi)$ (or $\tilde{u}_* \in (0, 1)$) satisfying

$$\nabla\lambda_1 \cdot r_1(\phi, u_*) = f_{uu}(\phi, u_*) = 0,$$
$$\nabla\lambda_1 \cdot r_1(\phi, u) = f_{uu}(\phi, u) > 0, \quad 0 < u < u_*,$$
$$\nabla\lambda_1 \cdot r_1(\phi, u) = f_{uu}(\phi, u) < 0, \quad u_* < u < \phi. \tag{2.18}$$

It implies that the 0th characteristic field is linear degenerate, but the first characteristic field is neither genuinely nonlinear or linear degenerate. That is, system (2.13) is non-strictly hyperbolic. Therefore, the Lax method for (2.15) and the Glimm method for (2.14) have to be modified, which will be carried out as follows.

To start, we construct the self-similar type of (generalized) weak solutions to (2.15) by Lax method. It requires to study the elementary waves for each wave field and the corresponding wave curves in ϕ-u space. We first construct the elementary waves in 0th characteristic field. Due to the linear degeneracy of characteristic field, the contact discontinuity connecting two constant states occurs in the solution of (2.15). Let $\{(\phi, u)\}$ be the states of such contact discontinuity connected to (ϕ_L, u_L), which is on the left of (ϕ, u). Then, following Rankine-Hugoniot condition and that the speed of jump is zero, we have

$$f(\phi, u) = f(\phi_L, u_L). \tag{2.19}$$

That is, (ϕ, u) satisfies

$$\frac{u^2}{u^2 + M(\phi - u)^2} = \frac{u_L^2}{u_L^2 + M(\phi_L - u_L)^2}. \tag{2.20}$$

Solving (2.20) directly, we obtain that states $\{(\phi, u)\}$ satisfy

$$u = \frac{u_L}{\phi_L}\phi. \tag{2.21}$$

It means that $\{(\phi, u)\}$ of contact discontinuity form a straight line passing through (ϕ_L, u_L) and crossing the line of critical state $u = u_*$ in $0 < u \leq \phi$. In particular, these states lie on the boundary $u = \phi$ when $u_L = \phi_L$. In addition, by plugging $u = \phi\tilde{u}$, $u_L = \phi_L\tilde{u}_L$ into (2.21), we obtain $\tilde{u} = \tilde{u}_L$ for the contact discontinuity, which means that there is no jump in $\tilde{u}$. Indeed, the jump of u is caused purely by the discontinuity of ϕ, but not by $\tilde{u}$.

Next, due to the non-convexity of f, we construct the generalized version of elementary waves for the first characteristic field. It can be accomplished by the convex extension of f in (2.9), see Ref. 27. By (2.16) we have $\lambda_1(\phi, u) > 0$ for $0 < u < \phi$ so that the generalized elementary wave is located in $x > 0, t > 0$, which implies that $\phi = \phi_R$ in (2.15). So, the generalized elementary wave solves

$$u_t + f(\phi_R, u)_x = 0. \tag{2.22}$$

Let (ϕ_L, u_L), (ϕ_R, u_R) be, respectively, the left and right constant states separated by the generalized elementary wave $U_1(\xi) = (\phi_R, u_1(\xi))$ where $\xi := \frac{x}{t}$, also let

$$\tilde{u}_L := \frac{u_L}{\phi_R}, \quad \tilde{u}_R := \frac{u_R}{\phi_R}, \quad \tilde{u}_1(\xi) := \frac{u_1(\xi)}{\phi_R}. \tag{2.23}$$

Then, by the case studies of $\tilde{u}_L$, $\tilde{u}_R$ and the results of Ref. 27, we have the following theorem for $\tilde{u}_1(\xi)$.

Theorem 2.1: *Consider Riemann problem (2.15). Suppose (ϕ_R, u_*) is the root of $f_{uu} = 0$. Define $\tilde{u}_* := \frac{u_*}{\phi_R}$. Then*

(i) If $0 < \tilde{u}_L < \tilde{u}_R \le \tilde{u}_*$ or $\tilde{u}_* \le \tilde{u}_R < \tilde{u}_L \le 1$, then $\tilde{u}_1(\xi)$ in (2.23) is a rarefaction wave connecting $\tilde{u}_L$, $\tilde{u}_R$ and satisfies

$$f_u(\phi_R, \tilde{u}_1(\xi)) = \xi. \tag{2.24}$$

(ii) If $0 < \tilde{u}_R < \tilde{u}_L \le \tilde{u}_*$ or $\tilde{u}_* \le \tilde{u}_L < \tilde{u}_R \le 1$, then $\tilde{u}_1(\xi)$ is a shock wave connecting $\tilde{u}_L$, $\tilde{u}_R$ and satisfies Rankine-Hugoniot jump condition

$$s(u_L - u_R) = f(\phi_R, u_L) - f(\phi_R, u_R), \tag{2.25}$$

where s is the speed of shock satisfying Lax entropy condition

$$f_u(\phi_R, u_R) < s < f_u(\phi_R, u_L). \tag{2.26}$$

(iii) If $0 < \tilde{u}_L < \tilde{u}_* < \tilde{u}_R \le 1$ or $0 < \tilde{u}_R < \tilde{u}_* < \tilde{u}_L \le 1$, then $\tilde{u}_1(\xi)$ is a rarefaction-shock connecting $\tilde{u}_L$, $\tilde{u}_R$. More precisely, for $0 < \tilde{u}_L < \tilde{u}_* < \tilde{u}_R \le 1$,

$$\tilde{u}_1(\xi) = \begin{cases} \tilde{u}_L, & \xi < f_u(\phi_R, u_L), \\ v(\xi), & f_u(\phi_R, u_L) \le \xi < f_u(\phi_R, \bar{u}), \\ \tilde{u}_R, & \xi > f_u(\phi_R, \bar{u}), \end{cases} \tag{2.27}$$

where $v(\xi)$ satisfies (2.24), $\bar{u} \in (0, 1)$ and satisfies

$$f_u(\phi_R, \phi_R \bar{u}) = \frac{f(\phi_R, u_R) - f(\phi_R, \phi_R \bar{u})}{u_R - \phi_R \bar{u}}. \tag{2.28}$$

For $0 < \tilde{u}_R < \tilde{u}_* < \tilde{u}_L \le 1$, $\tilde{u}_1(\xi)$ is obtained by switching $\tilde{u}_L$ with $\tilde{u}_R$ and u_L with u_R in (2.27). Furthermore, $\tilde{u}_1(\xi)$ in all the cases above are monotone in ξ.

The entropy condition imposed in Theorem 2.1, which combines the convex extension of f with Lax's entropy condition, selects the entropy solution of (2.15) in the sense that, the admissible elementary wave of (2.15) has least total variation and preserves the monotonicity of its profile.

Proof: To show (i) and (ii), since u_L and u_R both lie on either the convex or concave portion of $f(\phi_R, u)$, $\tilde{u}_1(\xi)$ and its monotonicity in ξ can be obtained as in Refs. 25 and 26. For (iii), since $v(\xi)$ in (2.27) satisfies (2.24), $v(\xi)$ is a rarefaction wave. In addition, constant states $\bar{u}$, $\tilde{u}_R$ satisfy Rankine-Hugoniot condition (2.28), which means that $\tilde{u}_1$ is a shock with speed $f_u(\phi_R, \phi_R \bar{u})$. It follows that $\tilde{u}_1(\xi)$ in (2.27) is a rarefaction-shock satisfying the definition of weak solutions to (2.15). It is easy to see that $\tilde{u}_1(\xi)$ in all cases above are monotone in ξ. We complete the proof.

We notice that the wave curve of first characteristic field passing through (ϕ_L, u_L) is the straight line $\phi = \phi_L$. It follows that the wave curves passing through (ϕ_L, u_L), which are $u = \frac{u_L}{\phi_L}\phi$ (0th characteristic field) and $\phi = \phi_L$ (first characteristic field), divide (ϕ, u) plane into four regions. Furthermore, under the transformation $(\phi, u) \to (\phi, \tilde{u})$, the wave curves above in (ϕ, u) plane can be transformed into the following simpler forms in $(\phi, \tilde{u})$ plane:

$$\begin{aligned} \tilde{u} &= \tilde{u}_L \quad \text{(0-th characteristic field)}, \\ \phi &= \phi_L \quad \text{(1st characteristic field)}. \end{aligned} \tag{2.29}$$

So, the weak solutions of (2.15) can be constructed by the case studies of (ϕ_R, u_R) or $(\phi_R, \tilde{u}_R)$ in each region. Therefore, by Theorem 2.1 we obtain the existence and uniqueness of weak solution u (or $\tilde{u}$) to (2.15).

Theorem 2.2: *Consider Riemann problem (2.15). Suppose $U_L = (\phi_L, u_L)$, $U_R = (\phi_R, u_R) \in \Omega$, and $|U_L - U_R|$ is sufficiently small. Then, under the Lax entropy condition and convex extension of flux f, there exists a neighborhood $N \subset \Omega$ such that if $U_L, U_R \in N$, then (2.15) admits a unique weak solution. The weak solution consists of at most three constant states separated by the rarefaction wave, shock, rarefaction-shock, or contact discontinuity. Moreover, $\tilde{u}$ in the weak solution is monotone in ξ.*

B. The Cauchy problem of zero order generalized BL equation

In this subsection, we establish the global existence of weak solutions to Cauchy problem (2.14) (or (2.10)) by the Glimm method. The Glimm method[16] employs the following steps:

(I) The construction of approximate solutions for (2.14);
(II) wave interaction estimate;
(III) compactness of the subsequence of approximate solutions; and
(IV) showing the limit of approximate solution is indeed a weak solution.

In this problem, we use the weak solutions of Riemann problem (2.15) as the building block of Glimm scheme. Then, by Theorem 2.2 and the Glimm scheme in Ref. 16, we are able to construct the sequence of approximate solutions $\{U_{\theta,\,\Delta x}\}$ for (2.14) where Δx is the grid size and $\theta := (\theta_1, \theta_2, \cdots)$ is a sequence of equi-distributed random numbers in $[-1, 1]$. Here we notice that the solution u of Riemann problem (2.15) may not be monotone in ξ, but $\tilde{u} = \frac{u}{\phi}$ is monotone in ξ. So, instead we construct the sequence of approximate solutions $\{\tilde{U}_{\theta,\Delta x} := (\phi_{\theta,\Delta x}, \tilde{u}_{\theta,\Delta x})\}$ for (2.14). To obtain the convergence of $\{\tilde{U}_{\theta,\Delta x}\}$, we need to obtain the uniform bound of total variation of $\{\tilde{U}_{\theta,\Delta x}\}$, which can be accomplished by controlling the difference of total variations between the in-coming and out-going waves of $\{\tilde{U}_{\theta,\Delta x}\}$ in each diamond region.[16] In our case, thanks to (2.29) we can simply use the case studies to observe that, the total variations of $\{\tilde{u}_{\theta,\Delta x}\}$ between the in-coming and out-going waves in each diamond region are identical. So, the BV-norm of $\{\tilde{U}_{\theta,\Delta x}\}$ is uniformly bounded if the total variation of $\tilde{U}_0(x) = (\phi_0(x), \tilde{u}_0(x))$ is bounded. It follows that $\{\tilde{U}_{\theta,\Delta x}\}$ is well defined for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$. Moreover, by the analysis in Ref. 36, we can show $\{\tilde{U}_{\theta,\Delta x}\}$ satisfies the following properties.

Theorem 2.3: *Suppose the total variation of $\tilde{U}_0(x)$ is small, then there exist constants $\{C_i\}_{i=1}^3$ which are independent of θ and Δx such that*

(i) $T.V.(\tilde{U}_{\theta,\Delta x}) \le C_1\, T.V.(\tilde{U}_0)$,
(ii) $T.V.[\tilde{U}_{\theta,\Delta x}(x, i\Delta t)] + \sup_x[\tilde{U}_{\theta,\Delta x}(x, i\Delta t)] < C_2\, T.V.(\tilde{U}_0)$,
(iii) $\int_{\mathbb{R}} |\tilde{U}_{\theta,\Delta x}(x, t_2) - \tilde{U}_{\theta,\Delta x}(x, t_1)| dx \le C_3\, (|t_2 - t_1| + \Delta t)$,

where $T.V.(U)$ denotes the total variation of U.

Therefore, by Theorem 2.3 and Oleinik's analysis,[16,33,36] we obtain the compactness of the subsequence of $\{\tilde{U}_{\theta,\Delta x}\}$, which means that there exists a subsequence $\{\tilde{U}_{\theta,\Delta x_i}\}$ of $\{\tilde{U}_{\theta,\Delta x}\}$ such that $\{\tilde{U}_{\theta,\Delta x_i}\}$ converges to some measurable function $\tilde{U}(x, t) = (\phi(x), \tilde{u}(x, t))$ in L^1_{loc}. In particular, by the triangle inequality we obtain $\phi_{\theta,\Delta x_i} \cdot \tilde{u}_{\theta,\Delta x_i} \to \phi \cdot \tilde{u}$ in L^1_{loc}. That is, $\{U_{\theta,\Delta x_i}\}$ converges to some measurable function U(x, t) = (ϕ(x), u(x, t)) in L^1_{loc} where $u(x, t) = \phi(x, t)\tilde{u}(x, t)$. In addition, $F(U_{\theta,\Delta x_i}) \to F(U)$ in L^1_{loc} as $\Delta x_i \to 0$.

Next, we show u is indeed a weak solution of (2.10), or equivalently U is a weak solution of (2.14), we calculate residual $R_\psi(U_{\theta,\,\Delta x})$ defined in (2.12) for any test function ψ. Following the analogous calculation of $R_\psi(U_{\theta,\,\Delta x})$ in Refs. 16 and 36, we obtain that $R_\psi(U_{\theta,\,\Delta x}) \to 0$ as $\Delta x \to 0$ in the probability space Φ of random numbers $\{\theta_j\}$. It implies that there is a null set $N \in \Phi$

and a sequence $\Delta x_i \to 0$ such that for any $\theta \in \Phi \backslash N$ and test function ψ, $R_\psi(U_{\theta,\,\Delta x}) \to 0$ as $\Delta x \to 0$. Therefore, U is a weak solution of (2.14). We then establish the global existence of weak solution to (2.10), which is given in the following theorem.

Theorem 2.4: *Consider the Cauchy problem* (2.10). *Suppose $T.V.(u_0)$ is sufficiently small. Then there exists a null set $N \in \Phi$ and a sequence $\Delta x_i \to 0$ such that if $\theta \in \Phi\backslash N$, $u := \lim_{\Delta x_i \to 0} u_{\theta, \Delta x_i}$ is a solution of* (2.10).

Next, we study the following initial-boundary value problem:

$$\begin{cases} u_t + f(\phi, u)_x = 0, & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & x > 0, \\ u(0,t) = u_B(t), & t > 0, \end{cases} \tag{2.30}$$

where $u_0(x)$ and $u_B(t)$ are of finite total variation. By definition $u \in L^1$ is a weak solution of (2.30) if

$$\iint_{x,\,t>0} u\psi_t + f(\phi,u)\psi_x dxdt + \int_0^\infty u_0(x)\psi(x,0)dx + \int_0^\infty f(\phi(0), u_B(t))\psi(0,t)dt = 0.$$

It is important to study the weak solution of the following boundary Riemann problem:

$$\begin{cases} U_t + F(U)_x = 0, & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ U(x,0) = (\phi_R, u_R)^T := U_R, & x > 0, \\ U(0,t) = (\phi_L, u_L)^T := U_L, & t > 0. \end{cases} \tag{2.31}$$

Note that, since 0th characteristic field is characteristic to boundary $x = 0$, there exists a contact discontinuity attached to the boundary in the solution of (2.31) if $\phi_L \neq \phi_R$. Following the results of Refs. 21 and 32, we are able to construct the weak solution of (2.31). The structure of solution of (2.31) is similar to that of (2.15). Furthermore, since no waves-reflection occurs on the boundary due to $\lambda_1 > 0$, the wave interaction estimate is similar to the one for Cauchy problem. Therefore, Glimm method can be applied analogously to (2.30), and we obtain the global existence of weak solutions to (2.30). We refer Refs. 21 and 32 for more details of Glimm method to initial-boundary value problems.

C. The Cauchy problem of first order generalized BL equation

In this subsection, we generalize the Glimm method to the following Cauchy and initial-boundary value problems of first order generalized BL equation:

$$\begin{cases} u_t + \tilde{F}(\varepsilon, t, x, \phi, u)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.32}$$

$$\begin{cases} u_t + \tilde{F}(\varepsilon, t, x, \phi, u)_x = 0, & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & x > 0, \\ u(0,t) = u_B(t), & t > 0, \end{cases} \tag{2.33}$$

where $\tilde{F}(\varepsilon, t, x, \phi, u)$ is given in (2.8). In view of (2.12), the weak solution u of (2.32) satisfies

$$R_\psi(\phi, u) = \iint_{t>0} u\psi_t + \tilde{F}(\varepsilon, t, x, \phi, u)\psi_x dxdt + \int_{-\infty}^\infty u_0(x)\psi(x,0)dx = 0, \tag{2.34}$$

for any $\psi \in C_0^\infty(\mathbb{R}\times[0,\infty))$. Similarly, u is a weak solution of (2.33) if u satisfies

$$\iint_{x,\,t>0} u\psi_t + \tilde{F}(\varepsilon,t,x,\phi,u)\psi_x dxdt + \int_0^\infty u_0(x)\psi(x,0)dx + \int_0^\infty \tilde{F}(\varepsilon,t,0,\phi(0),u_B(t))\psi(0,t)dt = 0.$$

To extend Glimm's method to (2.32), we first freeze variables x, t in $\tilde{F}(\varepsilon,t,x,\phi,u)$ in each grid and study the following modified Riemann problem with center at grid point (x_0, t_0):

$$\begin{cases} U_t + \bar{F}(\varepsilon,t_0,x_0,U)_x = 0, & x\in[x_0-\Delta x, x_0+\Delta x]\times(t_0,t_0+\Delta t),\\ U(x,t_0) = \begin{cases} U_L := (\phi_L,u_L)^T, & x_0-\Delta x < x < x_0,\\ U_R := (\phi_R,u_R)^T, & x_0 < x < x_0+\Delta x,\end{cases} \end{cases} \tag{2.35}$$

where $U=(\phi,u)^T$ and

$$\bar{F}(\varepsilon,t_0,x_0,U) = (0,\tilde{F}(\varepsilon,t_0,x_0,\phi,u))^T = \Big(0, f(\phi,u)\big(1+\varepsilon h(x_0,t_0)(1-\frac{u}{\phi})^2\big)\Big)^T. \tag{2.36}$$

We first study the weak solution of (2.35) by Lax's method. We have the following eigenvalues and the corresponding eigenvectors of Jacobian $D\bar{F}$:

$$\begin{aligned} \bar{\lambda}_0(\varepsilon,x_0,t_0,U) &= 0,\\ \bar{\lambda}_1(\varepsilon,x_0,t_0,U) &= \tilde{F}_u(t_0,x_0,\phi,u) = f_u(U) + \varepsilon\eta_1(x_0,t_0,U),\\ \bar{r}_0(\varepsilon,x_0,t_0,U) &= (\tilde{F}_u, -\tilde{F}_\phi)^T = (f_u,-f_\phi)^T + \varepsilon(\eta_1(x_0,t_0,U),\eta_2(x_0,t_0,U))^T,\\ \bar{r}_1(\varepsilon,x_0,t_0,U) &= (0,1), \end{aligned} \tag{2.37}$$

where

$$\eta_1(x_0,t_0,U) := h(x_0,t_0)\frac{\partial}{\partial u}\big(f(\phi,u)(1-\frac{u}{\phi})^2\big),$$

$$\eta_2(x_0,t_0,U) := h(x_0,t_0)\frac{\partial}{\partial \phi}\big(f(\phi,u)(1-\frac{u}{\phi})^2\big).$$

Note that $\eta_1(x_0,t_0,U)$ and $\eta_2(x_0,t_0,U)$ are bounded functions for every $(x_0,t_0)\in\mathbb{R}\times\mathbb{R}^+$. So $\bar{\lambda}_1(\varepsilon,x_0,t_0,U)$ and $\bar{r}_0(\varepsilon,x_0,t_0,U)$ are the regular perturbations of $\lambda_1(U)$, $r_0(U)$ in (2.16), respectively. Therefore, by (2.17) and (2.18), it follows that for sufficiently small ε, there is a unique $\bar{u}\in(0,\phi)$ satisfying

$$\begin{aligned} \nabla\bar{\lambda}_0\cdot\bar{r}_0(\phi,u) &= 0, \quad 0<u<\phi,\\ \nabla\bar{\lambda}_1\cdot\bar{r}_1(\phi,\bar{u}) &= 0,\\ \nabla\bar{\lambda}_1\cdot\bar{r}_1(\phi,\bar{u}) &> 0, \quad 0<u<\bar{u},\\ \nabla\bar{\lambda}_1\cdot\bar{r}_1(\phi,\bar{u}) &< 0, \quad \bar{u}<u<\phi. \end{aligned} \tag{2.38}$$

It implies that $\tilde{F}(\varepsilon,t_0,x_0,\phi,u)$ is a convex-concave flux for every $(x_0,t_0)\in\mathbb{R}\times\mathbb{R}^+$, $0\le u<\phi$ and small $\varepsilon\ge 0$. So, the system in (2.35) is still non-strictly hyperbolic, and the 0th characteristic field is linear degenerate, while the first characteristic field is neither genuinely nonlinear nor linear degenerate.

Now we apply the Lax method to construct the weak solution of (2.35). First, we construct the wave curves for (2.35) in the region $0<u<\phi$. The states $\{(\phi,u)\}$ of 0th wave curve starting at (ϕ_L,u_L) satisfy

$$L(\varepsilon,t_0,x_0,\phi_L,u_L,\phi,u) = 0, \tag{2.39}$$

where $L(\varepsilon, \phi_L, u_L, \phi, u) = \tilde{F}(\varepsilon, t_0, x_0, \phi, u) - \tilde{F}(\varepsilon, t_0, x_0, \phi_L, u_L)$. Under the transformation $(\phi, u) \to (\phi, \tilde{u})$, $\tilde{u} = \frac{u}{\phi}$, we can re-write (2.39) into

$$H(\tilde{u}, \tilde{\varepsilon}) = \frac{\tilde{u}^2}{\tilde{u}^2 + M(1-\tilde{u})^2}\big(1 + \tilde{\varepsilon}(1-\tilde{u})^2\big) - \frac{\tilde{u}_L^2}{\tilde{u}_L^2 + M(1-\tilde{u}_L)^2}\big(1 + \tilde{\varepsilon}(1-\tilde{u}_L)^2\big) = 0, \tag{2.40}$$

where $\tilde{\varepsilon} := \varepsilon h(x_0, t_0)$. It is easy to see $\tilde{u} = \tilde{u}_L$ is a solution of (2.40). Moreover, for any fixed $\tilde{\varepsilon} \ll 1$, $H(\tilde{u}) = H(\tilde{u}, \tilde{\varepsilon})$ satisfies

$$H(0) < 0, \quad H(1) > 0, \tag{2.41}$$

$$H'(\tilde{u}, \tilde{\varepsilon}) = \tilde{f}'(\tilde{u})\big(1 + (1-\tilde{u})^2\big) - 2\tilde{\varepsilon}\tilde{f}(\tilde{u})(1-\tilde{u}),$$

where $\tilde{f}$ is as in (2.2), in particular, we have

$$H'(\tilde{u}, 0) = \tilde{f}'(\tilde{u})\big(1 + (1-\tilde{u})^2\big) > 0. \tag{2.42}$$

It follows by (2.41) and (2.42) that, for sufficiently small $\tilde{\varepsilon}$, $\tilde{u} = \tilde{u}_L$ is the unique solution of (2.40), or equivalently, $u = \frac{u_L}{\phi_L}\phi$ is the unique solution of (2.39). It follows that the 0th wave curve starting at (ϕ_L, u_L) can be expressed as $u = \frac{u_L}{\phi_L}\phi$ in (ϕ, u)-coordinate, or $\tilde{u} = \tilde{u}_L$ in $(\phi, \tilde{u})$-coordinate. Next, we study the 0th wave curves starting at the points on boundary $\phi = u$. The 0th wave curve starting at (ϕ_R, ϕ_R) satisfies $L(\epsilon, \phi_R, \phi_R, \phi, u) = 0$, or equivalently

$$f(\phi, u)\big(1 + \varepsilon h(x_0, t_0)(1 - \frac{u}{\phi})^2\big) = f(\phi_R, \phi_R)\big(1 + \varepsilon(1 - \frac{\phi_R}{\phi_R})^2\big) = 1. \tag{2.43}$$

By solving (2.43) directly, we obtain the following solutions of (2.43):

$$u = \frac{M}{\varepsilon h(x_0, t_0)}\phi, \quad u = \phi. \tag{2.44}$$

Under (2.6), the first solution in (2.44) is not admissible. It follows that the 0th wave curve starting at (ϕ_R, ϕ_R) is given by $u = \phi$. From previous analysis we see that the 0th wave curves for the first order generalized BL equation is identical to the ones in zero order case. To the first characteristic field, the generalized elementary wave satisfies

$$u_t + \tilde{F}(\varepsilon, t_0, x_0, \phi_R, u)_x = 0, \tag{2.45}$$

and the wave curve starting at (ϕ_R, u_L) is the straight line

$$\phi = \phi_R. \tag{2.46}$$

Next, from (2.38) we see that both $\tilde{F}(\varepsilon, t_0, x_0, \phi, u)$ and $f(\phi, u)$ are convex-concave functions of u. Indeed, $\tilde{F}(\varepsilon, t_0, x_0, \phi, u)$ is the regular perturbation of $f(\phi, u)$ for small $\varepsilon h(x_0, t_0)$. Therefore, from previous analysis we can obtain an analog result of Theorem 2.1 for the existence and uniqueness of weak solutions to (2.35).

Now, we use weak solution u of (2.35) as the approximate solution of the following original Riemann problem:

$$\begin{cases} U_t + \bar{F}(\varepsilon, t, x, U)_x = 0, & x \in [x_0 - \Delta x, x_0 + \Delta x] \times (t_0, t_0 + \Delta t), \\ U(x, t_0) = \begin{cases} U_L := (\phi_L, u_L)^T, & x_0 - \Delta x < x < x_0, \\ U_R := (\phi_R, u_R)^T, & x_0 < x < x_0 + \Delta x. \end{cases} \end{cases} \tag{2.47}$$

It is necessary to calculate the residual (error) caused by this approximation in each local region $\Omega(x_0, t_0) := \{|x - x_0| \le \Delta x, t_0 \le t \le t_0 + \Delta t\}$. Given weak solution u of (2.35), we define

$$\tilde{R}_\psi(\phi, u, \Omega(x_0, t_0)) := \iint_{\Omega(x_0, t_0)} u\psi_t + \tilde{F}(\varepsilon, t_0, x_0, \phi, u)\psi_x dxdt. \tag{2.48}$$

Then, using the divergence theorem and Rankine-Hugoniot condition, we obtain

$$\begin{aligned}\tilde{R}_\psi(\phi, u, \Omega(x_0, t_0)) = &\int_{x_0-\Delta x}^{x_0+\Delta x} u(x, t_0+\Delta t)\psi(x, t_0+\Delta t)dx \\ &- \int_{x_0-\Delta x}^{x_0+\Delta x} u(x, t_0)\psi(x, t_0)dx \\ &+ \int_{t_0}^{t_0+\Delta t} \tilde{F}(\varepsilon, t_0, x_0, \phi_R, u(x_0+\Delta x, t))\psi(x_0+\Delta x, t)dt \\ &- \int_{t_0}^{t_0+\Delta t} \tilde{F}(\varepsilon, t_0, x_0, \phi_L, u(x_0-\Delta x, t))\psi(x_0-\Delta x, t)dt, \end{aligned} \tag{2.49}$$

for every test function $\psi \in C_0^\infty(\mathbb{R} \times [0, \infty))$. On the other hand, define

$$R_\psi(\phi, u, \Omega(x_0, t_0)) := \iint_{\Omega(x_0,t_0)} u\psi_t + \tilde{F}(\varepsilon, t, x, \phi, u)\psi_x dxdt. \tag{2.50}$$

Then, by the fact $h(x, t) \in C^1$ and *CFL* condition, we obtain

$$\left|R_\psi(\phi, u, \Omega(x_0, t_0)) - \tilde{R}_\psi(\phi, u, \Omega(x_0, t_0))\right| = O(1)(\Delta x)^3. \tag{2.51}$$

It follows by (2.49) and (2.51) that

$$\begin{aligned}R_\psi(\phi, u, \Omega(x_0, t_0)) = &\int_{x_0-\Delta x}^{x_0+\Delta x} u(x, t_0+\Delta t)\phi(x, t_0+\Delta t)dx \\ &- \int_{x_0-\Delta x}^{x_0+\Delta x} u(x, t_0)\phi(x, t_0)dx \\ &+ \int_{t_0}^{t_0+\Delta t} \tilde{F}(\varepsilon, t_0, x_0, \phi_R, u(x_0+\Delta x, t))\phi(x_0+\Delta x, t)dt \\ &- \int_{t_0}^{t_0+\Delta t} \tilde{F}(\varepsilon, t_0, x_0, \phi_L, u(x_0-\Delta x, t))\phi(x_0-\Delta x, t)dt \\ &+ O(1)(\Delta x)^3. \end{aligned} \tag{2.52}$$

We will need (2.52) to obtain the consistency of generalized Glimm scheme.

We now establish the global existence of weak solutions to (2.32). The generalized Glimm scheme for the approximate solutions $\{u_{\theta,\Delta x}\}$ of (2.32) can be constructed similarly as the zero order case. We emphasize that, in contrast to the zero order case, in each time step we solve for different Riemann problems caused not only by the difference of initial data, but also by the difference of fluxes due to the choice of $h(x, t)$. To obtain the stability of generalized Glimm scheme, since the structure of wave curve is the same as the one in zero order case, we obtain the similar wave interaction estimates so that the total variation of $\{\tilde{u}_{\theta,\Delta x}\}$ is conserved in each diamond region. It leads to the analog result of Theorems 2.3 and the compactness of subsequence of $\{\tilde{u}_{\theta,\Delta x}\}$. For the consistency of scheme, we apply the divergence theorem and (2.52) to residual $R_\psi(\phi, u)$ in (2.34), and by the results in Ref. 36 to obtain

$$R_\psi(\phi_{\theta,\Delta x}, u_{\theta,\Delta x}) = \sum_{i+j=even} R_\psi(\phi_{\theta,\Delta x}, u_{\theta,\Delta x}, \Omega(x_i, t_j)) \to 0 \quad as\ \Delta x \to 0,$$

in the sense of probability space of equi-distributed random numbers $\{\theta_i\}$. We then obtain the consistency of scheme, as well as the global existence of weak solution to (2.32). The Glimm method to initial-boundary value problem (2.33) is similar to the one for zero order case, so we omit it. We then have the following theorem for the Cauchy and initial-boundary problems of first order BL equation.

Theorem 2.5: *Consider Cauchy problem (2.32) and initial-boundary value problem (2.33). Let $\{u_{\theta,\Delta x}\}$ and $\{\bar{u}_{\bar{\theta},\Delta x}\}$ denote, respectively, the approximate solutions of (2.32)* and *(2.33) by generalized Glimm scheme. Suppose (2.6) holds and the total variations of $u_0(x)$ and $u_B(t)$ are sufficiently small. Then there exist null sets $N_C, N_B \in \Phi$, and a sequence $\Delta x_i \to 0$ such that if $\theta \in \Phi \backslash N_C$ ($\bar{\theta} \in \Phi \backslash N_B$, respectively), then $u := \lim_{\Delta x_i \to 0} u_{\theta,\Delta x_i}$ ($\bar{u} := \lim_{\Delta x_i \to 0} \bar{u}_{\bar{\theta},\Delta x_i}$, respectively) is a solution of (2.32) ((2.33), respectively).*

III. REGULARIZED BL EQUATION

We now turn to the regularized BL equation (1.2). For the simplicity of presentation and without loss of generality, we set $\phi(x) \equiv 1$ and write $f(u)$ for $f(1, u)$. We establish the global (in time) existence and uniqueness of solutions to the Cauchy problem

$$\begin{cases} u_t + (f(u))_x = \nu u_{xx} + \gamma u_{xxt}, & x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

and to the IBVP,

$$\begin{cases} u_t + f(u)_x = \nu u_{xx} + \gamma u_{xxt}, & x \geq 0,\ t \geq 0, \\ u(x,0) = u_0(x), & x \geq 0, \\ u(0,t) = g(t), & t \geq 0. \end{cases} \tag{3.2}$$

Here, $\nu \geq 0$ and $\gamma \geq 0$ are real parameters with at least one of them nonzero. u_0 and g in (3.2) satisfy the compatibility condition,

$$u_0(0) = g(0).$$

In addition, we obtain the eventual periodicity of solutions to (3.2). We remark that these results are partially influenced by the papers of Bona and co-workers on the well posedness of the Cauchy problem and the IBVP for the KdV and BBM equations in various functional settings.[3–5,7,8,11] What is special about the regularized BL equation is the particular form of the nonlinear term.

The global solution result for (3.1) can be stated as follows.

Theorem 3.1: *Assume at least one of the parameters $\nu \geq 0$ and $\gamma \geq 0$ is positive. Let $u_0 \in H^k(\mathbb{R})$ with $k \geq 1$. Then (3.1) has a unique global solution u satisfying $C([0,\infty); H^k(\mathbb{R}))$. In addition, $u \in C^\infty((0,\infty) \times \mathbb{R})$.*

The IBVP (3.2) also has a unique global solution for sufficiently smooth data. Here, $C_b^j([0,\infty))$ denotes the bounded continuously differentiable functions with the derivatives up to the jth order bounded.

Theorem 3.2: *Consider the IBVP for the regularized BL equation (3.2) with $\gamma > 0$ and $\nu \geq 0$. Let $u_0 \in H^2([0,\infty)) \cap C_b^2([0,\infty))$. Let $T > 0$ be fixed and let $g \in C^1((0,T))$. Then (3.2) has a unique global solution satisfying $C([0,\infty); H^2([0,\infty)) \cap C_b^2([0,\infty)))$.*

In the case when $\nu > 0$ and $g(t)$ is periodic, the solution is eventually periodic. More precisely, we have the following theorem.

Theorem 3.3: *Let $\nu > 0$* and *$\gamma > 0$. Let $u_0 \in H^2([0,\infty)) \cap C_b^2([0,\infty))$. Assume that $g \in C^1((0,\infty))$ is periodic, namely, for some $T_0 > 0$,*

$$g(t + T_0) = g(t) \quad \text{for all } t \geq 0.$$

Then the corresponding solution of (3.2) is eventually periodic in the sense that, for any $x > 0$,

$$\lim_{t \to \infty} (u(x, t + T_0) - u(x,t)) = 0.$$

We now turn to the proofs of these results.

Proof of Theorem 3.1: The proof consists of two major parts. The first part proves the local existence and uniqueness through the contraction-mapping principle. The second part establishes the global *a priori* bounds for $\|u\|_{H^k}$ and its C^∞-smoothness. Since the first part is standard (see, e.g., Refs. 3 and 4), attention is focused on the global bounds.

To prove the global bounds, we start with the L^2-norm. Multiplying (3.1) by u and integrating over $\mathbb{R}$, we have

$$\frac{d}{dt}\left(\|u\|_{L^2}^2 + \gamma\|u_x\|_{L^2}^2\right) + 2\nu\|u_x\|_{L^2}^2 = 0,$$

where we have used the fact that

$$\int f'(u)uu_x\,dx = \int (G(u))_x\,dx = 0, \quad G(u) = \int f'(u)\,u\,du. \tag{3.3}$$

Therefore, for any $t \geq 0$,

$$\|u\|_{L^2}^2 + \gamma\|u_x\|_{L^2}^2 + 2\nu\int_0^t \|u_x\|_{L^2}^2\,d\tau = \|u_0\|_{L^2}^2 + \gamma\|u_{0x}\|_{L^2}^2. \tag{3.4}$$

To estimate the H^1-norm, we take the inner product of $\partial_x u$ with ∂_x of (3.1) to obtain

$$\frac{d}{dt}\left(\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2\right) + 2\nu\|u_{xx}\|_{L^2}^2 = I_1,$$

where

$$I_1 = -2\int f'(u)\,u_x\,u_{xx}\,dx - 2\int f''(u)\,u_x^3\,dx.$$

To bound I_1, we first notice that $f(u)$ and all of its derivatives are bounded, namely,

$$\|f^{(l)}(u)\|_{L^\infty} \leq C_l \quad \text{for any integer } l \geq 0 \tag{3.5}$$

and then divide into two cases. In the case when $\nu > 0$,

$$\begin{aligned}
|I_1| &\leq \frac{\nu}{2}\|u_{xx}\|_{L^2}^2 + C\int (f'(u))^2\,u_x^2\,dx + C\,\|u_x\|_{L^\infty}\,\|u_x\|_{L^2}^2 \\
&\leq \frac{\nu}{2}\|u_{xx}\|_{L^2}^2 + C\,\|u_x\|_{L^2}^2 + C\,\|u_x\|_{L^2}^{5/2}\,\|u_{xx}\|_{L^2}^{1/2} \\
&\leq \nu\|u_{xx}\|_{L^2}^2 + C\,\|u_x\|_{L^2}^2 + C\,\|u_x\|_{L^2}^{10/3}.
\end{aligned}$$

In the case when $\gamma > 0$,

$$\begin{aligned}
|I_1| &\leq C\,\left(\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2\right) + C\,\|u_x\|_{L^\infty}\,\|u_x\|_{L^2}^2 \\
&\leq C\,\left(\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2\right) + C\sqrt{\left(\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2\right)}\|u_x\|_{L^2}^2.
\end{aligned}$$

Therefore, when $\nu > 0$, we have

$$\frac{d}{dt}\left(\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2\right) + \nu\|u_{xx}\|_{L^2}^2 \leq C\,\|u_x\|_{L^2}^2\left(1 + \|u_x\|_{L^2}^{4/3}\right).$$

Noticing (3.4) and applying Gronwall's inequality, we obtain

$$\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2 + \nu\int_0^t \|u_{xx}\|_{L^2}^2\,d\tau \leq C(T, \|u_0\|_{H^2}) \tag{3.6}$$

for any $t \leq T$. When $\gamma > 0$, we have

$$\frac{d}{dt}\sqrt{\left(\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2\right)} \leq C\sqrt{\left(\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2\right)} + C\,\|u_x\|_{L^2}^2$$

which yields the desired bound (3.6) again. It is not hard to see that higher order norms can be obtained in a similar fashion. Therefore, we get $u \in L^\infty([0,\infty); H^k(\mathbb{R}))$. To see that $u \in C([0,\infty); H^k(\mathbb{R}))$, we need to use the integral form. By taking the Fourier transform, we write (3.1) in the following integral form:

$$u(x,t) = \int K(x-y,t)u_0(y)\,dy + \int_0^t \int G(x-y,t-\tau)\, f(u(y,\tau))\,dy\,d\tau,$$

where

$$K(x,t) = \int e^{ix\xi}\, e^{-\frac{\nu\xi^2}{1+\gamma\xi^2}t}\,d\xi,$$

$$G(x,t) = \int e^{ix\xi}\, e^{-\frac{\nu\xi^2}{1+\gamma\xi^2}t}\,\frac{i\xi}{1+\gamma\xi^2}\,d\xi.$$

By an inductive argument similar to Ref. 3, we can show that $u \in C^\infty((0,\infty)\times\mathbb{R})$.

We finally verify the uniqueness of the solutions. Let $u, \tilde{u} \in L^\infty([0,T]; H^1(\mathbb{R}))$ be two solutions. Their difference $v = u - \tilde{u}$ satisfies

$$v_t + (f(u) - f(\tilde{u}))_x = \nu v_{xx} + \gamma v_{xxt}.$$

Therefore,

$$\frac{d}{dt}\left(\|v\|_{L^2}^2 + \gamma\|v_x\|_{L^2}^2\right) + 2\nu\|v_x\|_{L^2}^2 = I_2, \tag{3.7}$$

where

$$I_2 = -\int (f(u) - f(\widetilde{u}))_x\, v\,dx.$$

Because of (3.5), $|f(u) - f(\widetilde{u})| \le C\,|v|$ and thus

$$|I_2| \le C\int |v|\,|v_x|\,dx \le \begin{cases} \nu\|v_x\|_{L^2}^2 + C\,\|v\|_{L^2}^2, & \text{when } \nu > 0,\\ C\left(\|v\|_{L^2}^2 + \gamma\|v_x\|_{L^2}^2\right) & \text{when } \gamma > 0.\end{cases}$$

Inserting the bounds above in (3.7) and applying Gronwall's inequality yield $v \equiv 0$ or $u = \widetilde{u}$.

In order to prove Theorem 3.2, we recall a representation. This formula can be derived in a similar fashion as in Refs. 12 and 22.

Lemma 3.4: If u solves the following IBVP:

$$\begin{cases} u_t + \alpha u_x - \nu u_{xx} + \lambda u - \gamma u_{xxt} = h(x,t), & x \ge 0,\ t \ge 0,\\ u(x,0) = u_0(x), & x \ge 0,\\ u(0,t) = g(t), & t \ge 0,\end{cases} \tag{3.8}$$

then u can be written as

$$u(x,t) = g(t)\,e^{-\frac{x}{\sqrt{\gamma}}} + \int_0^\infty \Gamma(x-y,t)\left[u_0(y) - g(0)\,e^{-\frac{y}{\sqrt{\gamma}}}\right]dy \tag{3.9}$$

$$+ \int_0^t\int_0^\infty \Phi(x-y,t-\tau)\left[h + \left(\frac{\alpha}{\sqrt{\gamma}} + \frac{\nu}{\gamma} - \lambda\right)g(\tau)\,e^{-\frac{y}{\sqrt{\gamma}}}\right]dy\,d\tau,$$

where Γ and Φ are given by

$$\Gamma(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\xi x}e^{-\delta t}e^{-i\beta t}\,d\xi,$$

$$\Phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{1}{1+\gamma\xi^2}\,e^{i\xi x}e^{-\delta t}e^{-i\beta t}\,d\xi$$

with

$$\delta = \frac{\lambda + \nu\xi^2}{1+\gamma\xi^2}, \quad \beta = \frac{\alpha\xi}{1+\gamma\xi^2}.$$

Proof of Theorem 3.2. The proof consists of two major parts. The first part establishes the local existence and uniqueness, while the second obtains the global bound.

To prove the local existence and uniqueness, we follow Lemma 3.4 to rewrite (3.2) into the integral form

$$u(x,t) = g(t)\, e^{-\frac{x}{\sqrt{\gamma}}} + \int_0^\infty \Gamma(x-y,t)\left[u_0(y) - g(0)\, e^{-\frac{y}{\sqrt{\gamma}}}\right] dy \tag{3.10}$$
$$+ \int_0^t \int_0^\infty \Phi(x-y,t-\tau)\left[-(f(u))_y + \frac{\nu g(\tau)}{\gamma}\, e^{-\frac{y}{\sqrt{\gamma}}}\right] dy d\tau,$$

where Γ and Φ are given by

$$\Gamma(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} e^{-\frac{\nu\xi^2}{1+\gamma\xi^2}t}\, d\xi, \qquad \Phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\gamma\xi^2}\, e^{i\xi x} e^{-\frac{\nu\xi^2}{1+\gamma\xi^2}t}\, d\xi.$$

It is then not hard to show that the right-hand side of (3.10) defines a contractive mapping from $C([0, T]; C_b)$ to itself when the time interval T is chosen to be sufficiently small. The details are similar to those in Ref. 5 or Ref. 11.

The global *a priori* bounds can be established following Ref. 11 although the nonlinear term is different. To be self-contained, we briefly outline the proof. Multiplying (3.2) by u and u_{xx}, respectively, and integrating over $(0, \infty)$, we obtain, after integrating by parts,

$$\frac{d}{dt}\left(\|u\|_{L^2}^2 + \gamma\|u_x\|_{L^2}^2\right) + 2\nu\|u_x\|_{L^2}^2$$
$$= -\int_0^\infty f'(u) u_x u\, dx - 2\nu g(t) u_x(0,t) - 2\gamma g(t) u_{xt}(0,t),$$
$$\frac{d}{dt}\left(\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2\right) + 2\nu\|u_{xx}\|_{L^2}^2$$
$$= -\int_0^\infty f''(u)\, u_x^3\, dx - 2g'(t) u_x(0,t) - \gamma f(g(t))\, u_x^2(0,t).$$

Adding the two equations above and integrating with respect to time, we have

$$\|u\|_{L^2}^2 + (1+\gamma)\|u_x\|_{L^2}^2 + \gamma\|u_{xx}\|_{L^2}^2 + \nu\int_0^t (\|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2)\, d\tau \tag{3.11}$$
$$= -\int_0^t\int_0^\infty (f'(u)u_x u + f''(u)\, u_x^3)\, dx - \int_0^t 2(\nu g(\tau) + g'(\tau)) u_x(0,\tau)\, d\tau$$
$$- \int_0^t (2\gamma g(\tau) u_{xt}(0,\tau) + \gamma f(g(\tau))\, u_x^2(0,\tau))\, d\tau.$$

The terms in the time integrals on the right can be bounded through Sobolev embedding inequalities. For example,

$$\left|\int_0^t g'(\tau) u_x(0,\tau)\, d\tau\right| \le C\, \|g\|_{L^\infty} \int_0^t \|u_x\|_{L^2}^{1/2} \|u_{xx}\|_{L^2}^{1/2}\, d\tau$$
$$\le C(T)\, \|g\|_{L^\infty}\left[\left(\int_0^t \|u_x\|_{L^2}^6\, d\tau\right)^{1/3} + \left(\int_0^t \|u_{xx}\|_{L^2}^2\, d\tau\right)^{1/3}\right].$$

Inserting these estimates in (3.11), we eventually obtain

$$C_1\, \|u\|_{H^1}^6 + C_2\|u_{xx}\|_{L^2}^2 \le C_3 + C_4\int_0^t \left[\|u\|_{H^1}^6 + \|u_{xx}\|_{L^2}^2\right] d\tau$$

and the desired bound then follows from Gronwall's inequality.

We now tend to the proof of Theorem 3.3. We need to evaluate the difference $u(x, t + T_0) - u(x, t)$. The following lemma provides a formula for this difference.

Lemma 3.5: If g is periodic in t with period T_0, then the solution u of (3.8) satisfies

$$\begin{aligned} &u(x,t+T_0)-u(x,t) \\ &\quad= \int_{-T_0}^{0}\int_0^\infty \Phi(x-y,t-\tau)\left[\overline{h}+(\frac{\alpha}{\sqrt{\gamma}}+\frac{\nu}{\gamma}-\lambda)g(\tau)\,e^{-\frac{y}{\sqrt{\gamma}}}\right]dy\,d\tau, \end{aligned} \tag{3.12}$$

where $\overline{h}$ is given by

$$\overline{h} = h-\alpha\,u_0'(x)+\nu u_0''(x)-\lambda u_0(x)-(\frac{\alpha}{\sqrt{\gamma}}+\frac{\nu}{\gamma}-\lambda)g(0)\,e^{-\frac{x}{\sqrt{\gamma}}}.$$

Proof: To verify the formula in the lemma, we consider the new function

$$v(x,t) = u(x,t)-u_0(x)+g(0)\,e^{-\frac{x}{\sqrt{\gamma}}} \tag{3.13}$$

and the equation it satisfies. Applying the formula in (3.9), we find that

$$\begin{aligned} v(x,t) &= g(t)\,e^{-\frac{x}{\sqrt{\gamma}}} \\ &\quad+\int_0^t\int_0^\infty \Phi(x-y,t-\tau)\left[\overline{h}+(\frac{\alpha}{\sqrt{\gamma}}+\frac{\nu}{\gamma}-\lambda)g(\tau)\,e^{-\frac{y}{\sqrt{\gamma}}}\right]dy\,d\tau. \end{aligned} \tag{3.14}$$

Clearly,

$$u(x,t+T_0)-u(x,t) = v(x,t+T_0)-v(x,t).$$

Using (3.14) and the fact that g is periodic, we have

$$\begin{aligned} &u(x,t+T_0)-u(x,t) \\ &\quad= \int_0^{t+T_0}\int_0^\infty \Phi(x-y,t+T_0-\tau)\left[\overline{h}+(\frac{\alpha}{\sqrt{\gamma}}+\frac{\nu}{\gamma}-\lambda)g(\tau)\,e^{-\frac{y}{\sqrt{\gamma}}}\right]dy\,d\tau \\ &\quad-\int_0^t\int_0^\infty \Phi(x-y,t-\tau)\left[\overline{h}+(\frac{\alpha}{\sqrt{\gamma}}+\frac{\nu}{\gamma}-\lambda)g(\tau)\,e^{-\frac{y}{\sqrt{\gamma}}}\right]dy\,d\tau. \end{aligned}$$

Making the substitution $\tau = T_0+\widetilde{\tau}$ in the first integral on the right and using the fact that $\overline{h}(x,t+T_0) = \overline{h}(x,t)$ and $g(t) = g(t+T_0)$, we obtain (3.12).

We now prove Theorem 3.3.

Proof of Theorem 3.3: Let u be a solution of (3.2). In the case when $\gamma > 0$, we use (3.12) to obtain

$$\begin{aligned} &u(x,t+T_0)-u(x,t) \\ &\quad= \int_{-T_0}^{0}\int_0^\infty \Phi(x-y,t-\tau)\left[-(f(u))_y+\nu u_0''(y)+\frac{\nu}{\gamma}(g(\tau)-g(0))\,e^{-\frac{y}{\sqrt{\gamma}}}\right]dy\,d\tau, \end{aligned} \tag{3.15}$$

where

$$\Phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\gamma\xi^2}\,e^{i\xi x}e^{-\frac{\nu\xi^2}{1+\gamma\xi^2}t}\,d\xi.$$

For $\nu > 0$,

$$|\Phi(x,t)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\gamma\xi^2}\,e^{-\frac{\nu\xi^2}{1+\gamma\xi^2}t}\,d\xi.$$

As $t\to\infty$,

$$\Phi(x,t)\to 0 \quad \text{uniformly in } x.$$

If $u \in C((0,\infty); H^2)$, then, by Sobolev's inequality

$$\int_0^\infty |f(u)_y|\,dy \le \int |f'(u)|\,|u_y|\,dy \le C\,\|u_x\|_{L^1}$$
$$\le C\,\|u\|_{L^2}^{3/4}\,\|u_{xx}\|_{L^2}^{1/4} \le C\,\|u\|_{H^2}.$$

Here, we have used the uniform boundedness of $f'(u)$. Letting $t \to \infty$ in (3.15) and applying the dominated convergence theorem yield

$$u(x, t + T_0) - u(x, t) \to 0.$$

This completes the proof of Theorem 3.3.

IV. CONVERGENCE

The goal of this section is to show that the solution of the regularized BL equation established in Theorem 3.1 converges to the corresponding ones of the generalized BL equation obtained in Theorem 2.4. We have the following theorem.

Theorem 4.1: *Let $u_0 \in H^2(\mathbb{R}) \cap C_b^2(\mathbb{R})$ and let $u^{\nu,\gamma}$ be the corresponding solution of (3.1) with $\phi(x) \equiv 1$. Then, if $\gamma = O(\nu^2)$, then $u^{\nu,\gamma}$ converges in L^p for $1 < p < 2$ to a weak solution u of the Cauchy problem for the generalized BL equation (2.10).*

Proof: We follow the approach of Schonbek.[35] Her approach was further developed and applied to diffusive-dispersive equations by LeFloch and Natalini[29] and LeFloch and Kondo.[28] The major idea of her approach is to show the boundedness of the entropy dissipation measures. That is, for any convex function ψ with ψ' and ψ'' uniformly bounded,

$$\partial_t \psi(u^{\nu,\gamma}) + \partial_x F(u^{\nu,\gamma})$$

is bounded when tested against a smooth test function, where $F' = \psi' f'$. More precisely, we prove that for any test function $\eta = \eta(x,t) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ with $\eta \ge 0$,

$$\int_0^\infty \int_{-\infty}^\infty (\partial_t \psi(u^{\nu,\gamma}) + \partial_x F(u^{\nu,\gamma}))\ \eta(x,t)\,dxdt$$

is bounded. It is easy to verify by multiplying (3.1) by ψ' that

$$\psi'(u^{\nu,\gamma})\partial_t u^{\nu,\gamma} + \psi'(u^{\nu,\gamma}) f' \partial_x u^{\nu,\gamma} = \psi'(u^{\nu,\gamma})\nu u_{xx}^{\nu,\gamma} + \gamma \psi'(u^{\nu,\gamma}) u_{xxt}^{\nu,\gamma}$$
$$= \nu(\psi'(u^{\nu,\gamma}) u_x^{\nu,\gamma})_x + \gamma(\psi'(u^{\nu,\gamma}) u_{xt}^{\nu,\gamma})_x - \nu\psi''(u^{\nu,\gamma})(u_x^{\nu,\gamma})^2 - \gamma\psi''(u^{\nu,\gamma}) u_x^{\nu,\gamma} u_{xt}^{\nu,\gamma}.$$

Therefore,

$$\int_0^\infty \int_{-\infty}^\infty (\partial_t \psi(u^{\nu,\gamma}) + \partial_x F(u^{\nu,\gamma}))\ \eta(x,t)\,dxdt = J_1 + J_2 + J_3 + J_4, \tag{4.1}$$

where, for the convenience of notation, we write $\widetilde{v}$ for $u^{\nu,\gamma}$,

$$J_1 = \int_0^\infty \int_{-\infty}^\infty \nu(\psi'(\widetilde{v})\widetilde{v}_x)_x\ \eta(x,t)\,dxdt,$$

$$J_2 = \int_0^\infty \int_{-\infty}^\infty \gamma(\psi'(\widetilde{v})\widetilde{v}_{xt})_x\ \eta(x,t)\,dxdt,$$

$$J_3 = -\int_0^\infty \int_{-\infty}^\infty \nu\psi''(\widetilde{v})\widetilde{v}_x^2\ \eta(x,t)\,dxdt,$$

$$J_4 = -\int_0^\infty \int_{-\infty}^\infty \gamma\psi''(\widetilde{v})\widetilde{v}_x\widetilde{v}_{xt}\ \eta(x,t)\,dxdt.$$

In order to bound the terms on the right, we need the following lemma.

Lemma 4.2: Let $u_0 \in H^2(\mathbb{R}) \cap C_b^2(\mathbb{R})$ and let $\widetilde{v}$ be the corresponding solution of (3.1) with $\nu > 0$ and $\gamma > 0$. Then $\widetilde{v}$ obeys the following bounds:

$$\|\widetilde{v}\|_{L^2}^2 + \gamma\|\widetilde{v}_x\|_{L^2}^2 + 2\nu\int_0^T \|\widetilde{v}_x\|_{L^2}^2\, d\tau = \|u_0\|_{L^2}^2 + \gamma\|u_{0x}\|_{L^2}^2. \tag{4.2}$$

$$\begin{aligned} &\frac{1}{2}\int \widetilde{v}^2\, dx + \nu\iint(\widetilde{v}_x^2 + 2\widetilde{v}_t^2)\, dxdt + \frac{1}{2}(2\nu^2+\gamma)\int \widetilde{v}_x^2\, dx \\ &\quad + 3\nu\iint \widetilde{v}_x^2\widetilde{v}^2\, dxdt + 2\nu\gamma\iint \widetilde{v}_{xt}^2\, dxdt = C_1\int u_0^2\, dx + C_2\int u_{0x}^2\, dx. \end{aligned} \tag{4.3}$$

To continue the proof of Theorem 4.1, we bound the terms on the right-hand side of (4.1). Integrating by parts, we have

$$|J_1| = \nu\left|\iint \psi'(\widetilde{v})\widetilde{v}_x\eta_x\, dxdt\right| \le \nu C\|\widetilde{v}_x\|_{L^2}\|\eta_x\|_{L^2} \le C\sqrt{\nu}\left(\nu\|\widetilde{v}_x\|_{L^2}^2 + \|\eta_x\|_{L^2}^2\right).$$

Integrating by parts and applying Hölder's inequality, we have

$$|J_2| = \gamma\left|\iint \psi'(\widetilde{v})\widetilde{v}_{xt}\eta_x\, dxdt\right| \le C\sqrt{\frac{\gamma}{\nu}}\left(\nu\gamma\iint \widetilde{v}_{xt}^2\, dxdt\right)^{\frac{1}{2}}.$$

By (4.3), J_2 is bounded when $\gamma = O(\nu)$. When $\gamma = o(\nu)$, $|J_2| \to 0$. According to (4.2),

$$|J_3| \le \frac{1}{2}\int \widetilde{v}_0^2(x)\, dx.$$

By Hölder's inequality,

$$|J_4| \le C\,\gamma^{1/2}\nu^{-1}\left(\nu\gamma\iint \widetilde{v}_{xt}^2\, dxdt\right)^{1/2}\left(\nu\iint \widetilde{v}_x^2\, dxdt\right)^{1/2}.$$

By (4.3) again, J_4 is bounded if $\gamma = O(\nu^2)$ and $J_4 \to 0$ if $\gamma = o(\nu^2)$. The desired convergence then follows from the scheme of Schonbek.[35]

We now turn to the proof of Lemma 4.2.

Proof of Lemma 4.2: The L^2-bound in (4.2) is obtained by multiplying (3.1) by $\widetilde{v}$ and integrating with respect to $(x, t) \in (-\infty, \infty) \times (0, \infty)$. Multiplying (3.1) by $\widetilde{v} + 2\nu\widetilde{v}_t$ and integrating with respect to $(x, t) \in (-\infty, \infty) \times (0, \infty)$, we obtain, after integration by parts,

$$\begin{aligned} &\frac{1}{2}\int \widetilde{v}^2\, dx + \nu\iint(\widetilde{v}_x^2 + 2\widetilde{v}_t^2)\, dxdt + \frac{1}{2}(2\nu^2+\gamma)\int \widetilde{v}_x^2\, dx \\ &\quad + 3\nu\iint \widetilde{v}_x^2\widetilde{v}^2\, dxdt + 2\nu\gamma\iint \widetilde{v}_{xt}^2\, dxdt \\ &= \frac{1}{2}\int u_0^2\, dx + \frac{1}{2}(2\nu^2+\gamma)\int u_{0x}^2\, dx - 2\nu\iint f'(\widetilde{v})\widetilde{v}_x\widetilde{v}_t\, dxdt. \end{aligned} \tag{4.4}$$

Here, we have used the fact that

$$\int \widetilde{v} f(\widetilde{v})_x dx = \int \widetilde{v} f'(\widetilde{v})\widetilde{v}_x dx = 0.$$

It suffices to bound the last term in the right-hand side of (4.4). Using the fact that

$$f'(\widetilde{v}) = \frac{2M\widetilde{v}(1-\widetilde{v})}{\widetilde{v}^2 + M(1-\widetilde{v})^2}$$

and $|f'(\widetilde{v})| \le C$, we have

$$2\nu\left|\iint f'(\widetilde{v})\widetilde{v}_x\widetilde{v}_t\, dxdt\right| \le C\nu\iint \widetilde{v}_x^2\, dxdt + \nu\iint \widetilde{v}_t^2\, dxdt.$$

Inserting this bound in (4.4), we obtain (4.3).

V. NUMERICAL RESULTS

This section presents numerical experiments performed on the BL and the regularized BL equations. These experiments are designed to understand the limiting behavior of solutions to the regularized BL equations as $\nu \to 0$ and $\gamma \to 0$. In particular, we examine how the relative sizes of ν and γ affect the convergence of solutions of the regularized BL equation to those of the BL equation.

We compute solutions of the BL equation

$$u_t + (f(u))_x = 0 \quad \text{with} \quad f(u) = \frac{u^2}{u^2 + M(1-u)^2} \tag{5.1}$$

and the regularized BL equation

$$u_t + (f(u))_x = \nu u_{xx} + \gamma u_{xxt} \tag{5.2}$$

corresponding to two initial data

$$u(x,0) = 1 - 0.4\exp(-10(x-5)^2), \quad 0 \le x \le 10, \tag{5.3}$$

and

$$u(x,0) = 1 - 0.9\exp(-10(x-5)^2), \quad 0 \le x \le 10. \tag{5.4}$$

We remark that we are computing the solutions of the Cauchy problem here. Let $u = 1 - w$. Then we can rewrite the above equation into the following Cauchy problem:

$$w_t + (f(w))_x = \nu w_{xx} + \gamma w_{xxt} \tag{5.5}$$

with $w(x,0) = 0.4\exp(-10(x-5)^2), 0.9\exp(-10(x-5)^2)$. We have truncated the computational domain from $\mathbb{R}$ to [0, 10]. Numerically, this is acceptable since the error due to the truncation can be made negligible. The numerical computations are done only for the above Cauchy problems. We consider $u(x, t)$ instead of $w(x, t)$ because $u(x, t)$ fits physical meanings better.

Without loss of generality, we take $M = 1$. Although (5.3) and (5.4) appear to be very similar, their corresponding solutions to (5.1) behave differently. To understand their behavior, we first plot the graphs of $f(u)$ and $f'(u)$ (see Figure 1), where

$$f'(u) = \frac{2u(1-2u)}{(u^2 + (1-u)^2)^2}.$$

For (5.3), $u_0(x) \ge 0.6$ and initially the propagation speed $f'(u)$ decreases as u increases. In contrast, the datum in (5.4) ranges between 0.1 and 1 and the initial propagation speed $f'(u)$ increases for u_0 in the range 0.1–0.5 but decreases for u_0 between 0.5 and 1. In fact, their corresponding numerical solutions of (5.1) are different, as we shall see below.

We now present our numerical results. First, we plot the solutions of (5.1) and (5.5) at $t = 1$ corresponding to the initial datum in (5.3). The following six graphs (see Figures 2–4) provide the solutions of (5.1) and of (5.5) with $\gamma = 0$ and $\nu = 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}$, and 0. These plots verify that, as $\nu \to 0$, the solution of (5.5) with $\gamma = 0$ tends to that of (5.1).

When $\gamma > 0$, if the condition $\gamma = O(\nu^2)$ is violated, the solution of (5.5) may not converge to that of (5.1), as the plots in Figure 5 indicate.

Now we present the numerical solutions of (5.1) and (5.5) at $t = 1$ corresponding to the initial datum in (5.4). The following six graphs (see Figures 6–8) provide the numerical solutions of (5.1) and of (5.5) with $\gamma = 0$ and $\nu = 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}$, and 0. Again these plots verify that, as $\nu \to 0$, the solution of (5.5) with $\gamma = 0$ tends to that of (5.1).

Again when $\gamma > 0$, if the condition $\gamma = O(\nu^2)$ is violated, the solution of (5.5) may not converge to that of (5.1) (see Figure 9).

Finally, we briefly indicate our numerical scheme. For the Cauchy problem we have used a numerical method developed in Ref. 4 with a fourth-order finite difference scheme for the spatial discretization ($\Delta x = 0.005$) and the fourth-order Runge-Kutta method for the temporal discretization

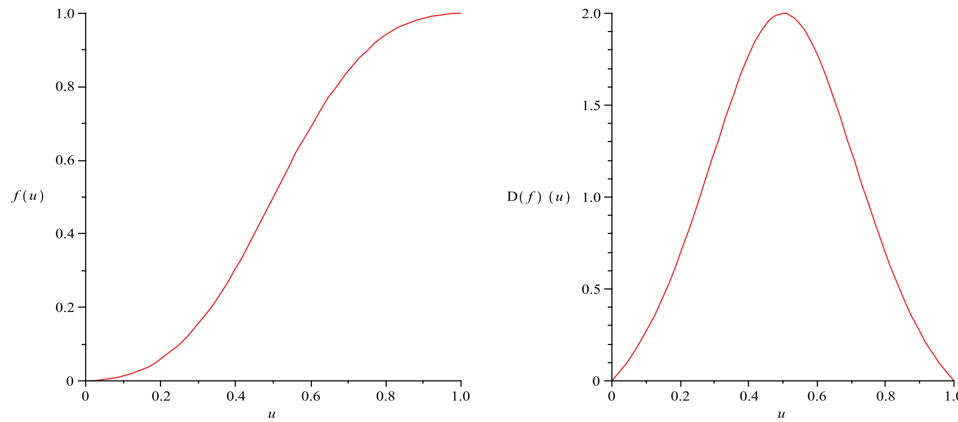

FIG. 1. The graphs of $f(u)$ and $f'(u)$.

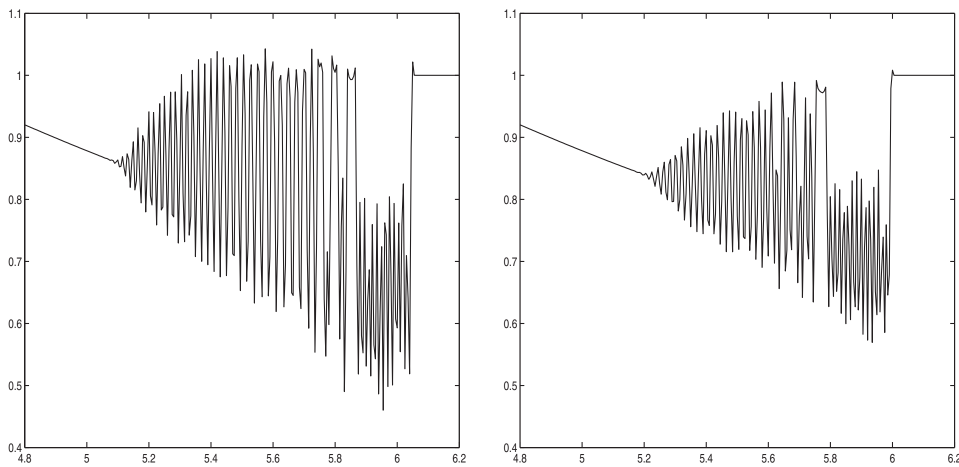

FIG. 2. The left represents the solution of (5.1), while the right the solution of (5.5) with $\gamma = 0$ and $\nu = 10^{-5}$ at time t = 1 with initial data in (5.3).

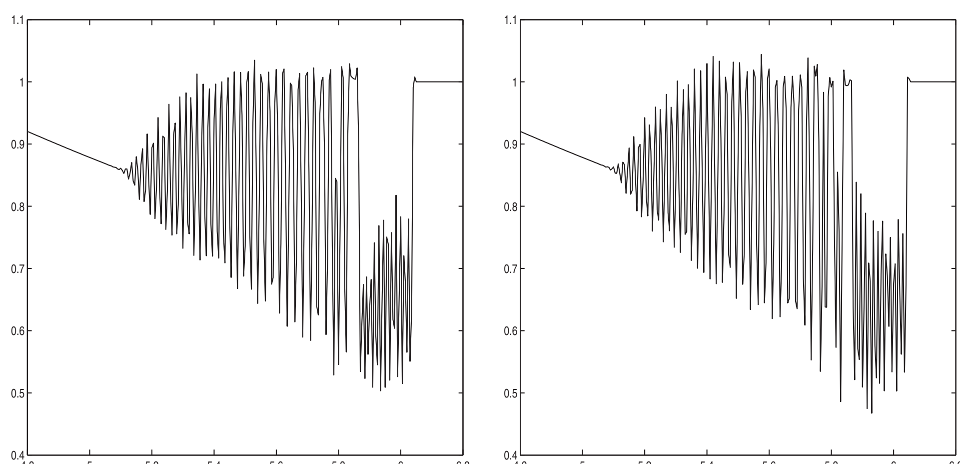

FIG. 3. The left represents the solution of (5.5) with $\gamma = 0$ and $\nu = 10^{-6}$, while the right that of (5.5) with $\gamma = 0$ and $\nu = 10^{-7}$ at time t = 1 with initial data in (5.3).

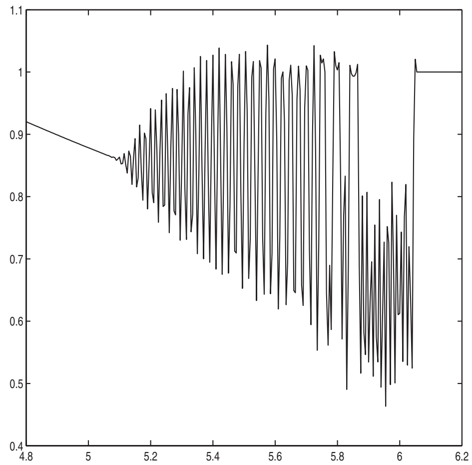

FIG. 4. The left represents the solution of (5.5) with $\gamma = 0$ and $\nu = 10^{-8}$, while the right that of (5.5) with $\gamma = 0$ and $\nu = 0$ at time t = 1 with initial data in (5.3).

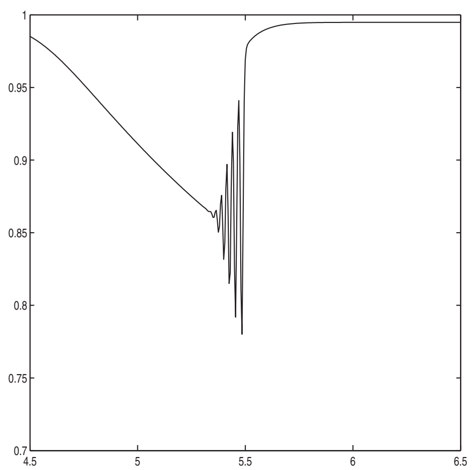

FIG. 5. The left represents the solution of (5.5) with $\gamma = 10^{-5}$ and $\nu = 10^{-5}$, while the right that of (5.1) at time t = 1 with initial data in (5.3).

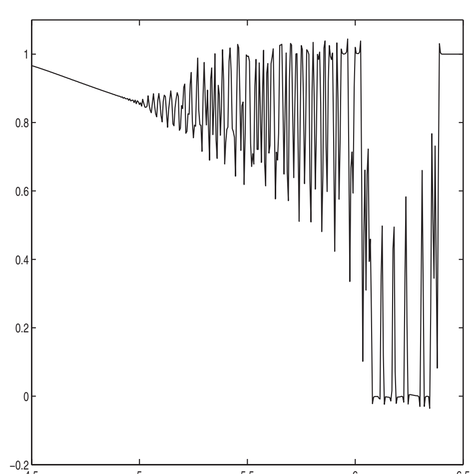

FIG. 6. The left represents the solution of (5.1), while the right the solution of (5.5) with $\gamma = 0$ and $\nu = 10^{-5}$ at time t = 1 with initial data in (5.4).

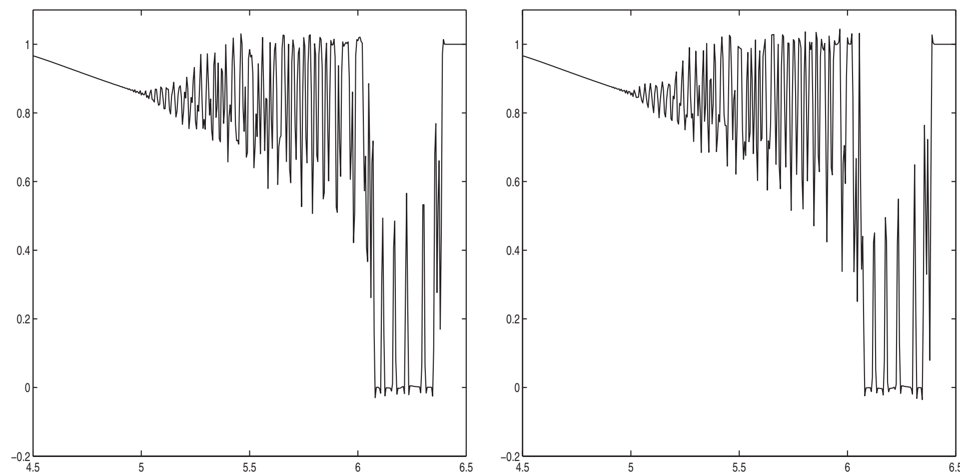

FIG. 7. The left represents the solution of (5.5) with $\gamma = 0$ and $\nu = 10^{-6}$, while the right that of (5.5) with $\gamma = 0$ and $\nu = 10^{-7}$ at time t = 1 with initial data in (5.4).

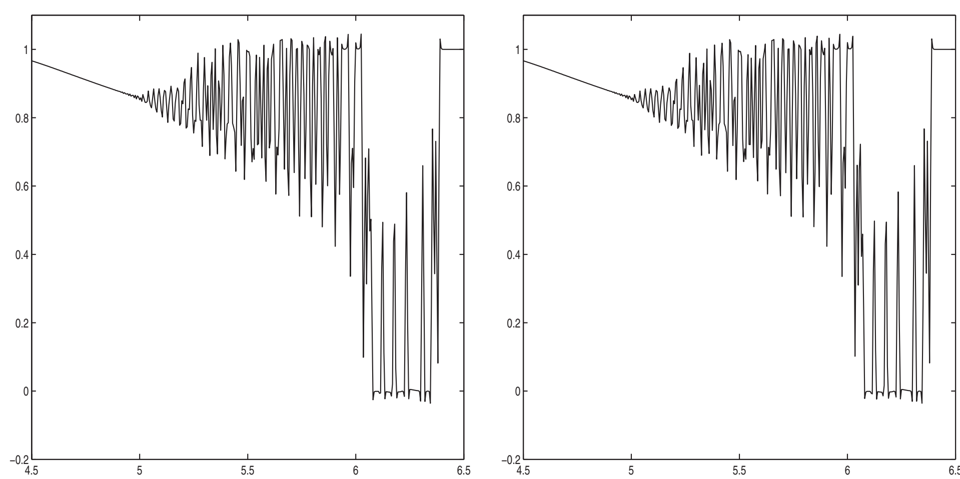

FIG. 8. The left represents the solution of (5.5) with $\gamma = 0$ and $\nu = 10^{-8}$, while the right that of (5.5) with $\gamma = 0$ and $\nu = 0$ at time t = 1 with initial data in (5.4).

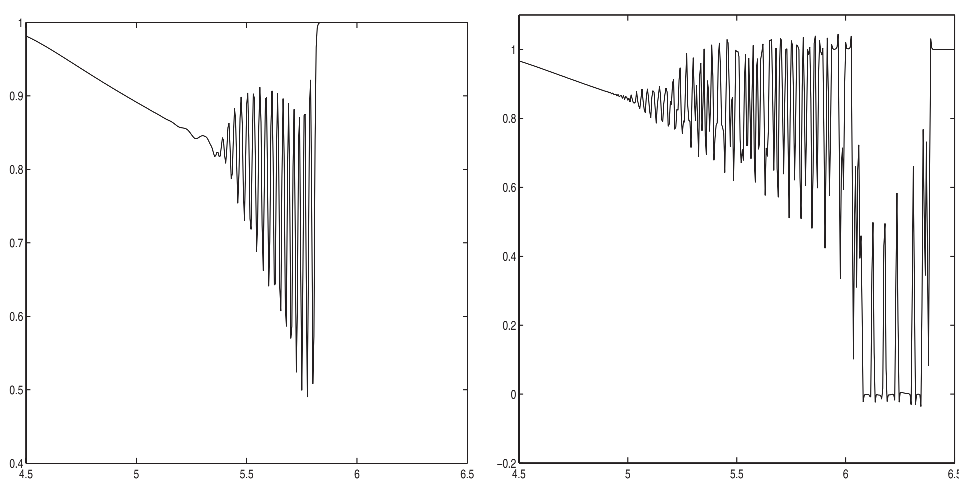

FIG. 9. The left represents the solution of (5.5) with $\gamma = 10^{-5}$ and $\nu = 10^{-5}$, while the right that of (5.1) at time t = 1 with initial data in (5.4).

($\Delta t = 10^{-5}$). For the IBVP of (5.5) with the boundary datum $g(t)$, we rewrite the equation into

$$u_t = g'(t)\,e^{-\frac{x}{\sqrt{\gamma}}} + \int_0^\infty \widetilde{K}(x,y)(f(u))(y,t)\,dy$$
$$+\frac{\nu}{\gamma}\Big[g(t)e^{-\frac{x}{\sqrt{\gamma}}} - u(x,t)\Big] - \nu\int_0^\infty \widetilde{H}(x,y)u(y,t)\,dy,$$

where $\widetilde{K}(x,y) = \frac{1}{2\gamma}[e^{-\frac{x+y}{\sqrt{\gamma}}} + sgn(x-y)e^{-\frac{|x-y|}{\sqrt{\gamma}}}]$ and $\widetilde{H} = \frac{1}{2\gamma^{\frac{3}{2}}}[e^{-\frac{x+y}{\sqrt{\gamma}}} + sgn(x-y)e^{-\frac{|x-y|}{\sqrt{\gamma}}}]$. Then the fourth-order Runge-Kutta method is used for the temporal discretization.

ACKNOWLEDGMENTS

Hong was partially supported by the National Science Council of the Republic of China under the Grant No. NSC-99-2115-M-008-006 and by the Center of Mathematics and Theoretical Physics at the National Central University. Wu was partially supported by NSF Grant No. DMS 0907913 and thanks the Mathematics Research Promotion Center, the Center of Mathematics and Theoretical Physics, and the Department of Mathematics at National Central University. Yuan was partially supported by the National Science Council of the Republic of China under the Grant No. NSC 99-2115-M-126-003. Yuan thanks Professor Goong Chen, Professor I-Liang Chern, and Professor Jyh-Hao Lee for their encouragements.

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