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# Global regularity results for the 2D Boussinesq equations with vertical dissipation

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ABSTRACT

This paper furthers the study of Adhikari et al. (2010) [2] on the global regularity issue concerning the 2D Boussinesq equations with vertical dissipation and vertical thermal diffusion. It is shown here that the vertical velocity $v$ of any classical solution in the Lebesgue space $L^q$ with $2 \leqslant q < \infty$ is bounded by $C_1 q$ for $C_1$ independent of $q$. This bound significantly improves the previous exponential bound. In addition, we prove that, if $v$ satisfies $\int_0^T \sup_{q\geqslant 2} \frac{\|v(\cdot,t)\|_{L^q}^2}{q}\,dt < \infty$, then the associated solution of the 2D Boussinesq equations preserve its smoothness on $[0,T]$. In particular, $\|v\|_{L^q} \leqslant C_2\sqrt{q}$ implies global regularity.



## 1. Introduction

This paper continues our study on the global regularity issue concerning the 2D Boussinesq equations with vertical dissipation and vertical thermal diffusion,

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu v_{yy} + \theta, \\ u_x + v_y = 0, \\ \theta_t + u\theta_x + v\theta_y = \kappa\theta_{yy}, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y), \quad \theta(x,y,0) = \theta_0(x,y), \end{cases} \tag{1.1}$$

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where $u, v, p$ and $\theta$ are scalar functions of $(x, y) \in \mathbf{R}^2$ and $t \geqslant 0$, $\nu > 0$ and $\kappa > 0$ are parameters. (1.1) is a very important special case of the full 2D Boussinesq equations

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu_1 u_{xx} + \nu_2 u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu_1 v_{xx} + \nu_2 v_{yy} + \theta, \\ u_x + v_y = 0, \\ \theta_t + u\theta_x + v\theta_y = \kappa_1 \theta_{xx} + \kappa_2 \theta_{yy}. \end{cases} \tag{1.2}$$

The Boussinesq equations model buoyancy-driven flows such as atmospheric fronts and oceanic circulation (see e.g. [19,21]). In these equations $u$ and $v$ denote the horizontal and the vertical velocity, respectively, $p$ the pressure, $\theta$ the temperature in the content of thermal convection and the density in the modeling of geophysical fluids. The Boussinesq equations with only vertical dissipation are useful in modeling dynamics of geophysical flows for which the vertical dissipation dominates such as in the large-time dynamics of certain strongly stratified flows (see [18] and the references therein).

One fundamental issue concerning the 2D Boussinesq equations (1.2) is whether all of their classical solutions are global in time. When the parameters $\nu_1$, $\nu_2$, $\kappa_1$ and $\kappa_2$ are all positive, this issue is not very difficult to resolve and any sufficiently smooth data leads to a global solution (see e.g. [5]). In the case of inviscid Boussinesq equations, namely (1.2) with $\nu_1 = \nu_2 = \kappa_1 = \kappa_2 = 0$, the global regularity problem turns out to be extremely difficult and remains outstandingly open. Important progress has recently been made on the intermediate cases. The global regularity for the case $\nu_1 = \nu_2 > 0$ and $\kappa_1 = \kappa_2 = 0$ was proven by Chae [7] and by Hou and Li [16]. The case when $\nu_1 = \nu_2 = 0$ and $\kappa_1 = \kappa_2 > 0$ was dealt with in [7]. Further progress on these two cases was made recently by Hmidi and Keraani, who were able to establish the global regularity with the full Laplacian operator $-\Delta$ replaced by $\sqrt{-\Delta}$ [14,15]. Danchin and Paicu very recently explored the global regularity issue for the cases when there is either horizontal dissipation ($\nu_1 > 0$ and $\nu_2 = \kappa_1 = \kappa_2 = 0$) or horizontal thermal diffusion ($\kappa_1 > 0$ and $\nu_1 = \nu_2 = \kappa_2 = 0$) and obtained global solutions at several regularity levels (see [11]). Other interesting recent results on the 2D Boussinesq equations can be found in [1,9,10,12,13, 17,20,22].

The global regularity problem for the Boussinesq equations with vertical dissipation and thermal diffusion, namely (1.1), was first studied by Adhikari et al. in [2]. As pointed out in [2], this is an extremely difficult problem. One main reason is that we have no global (in time) bound for any Sobolev norm of the solutions. As we can see from the equation for the vorticity $\omega = v_x - u_y$,

$$\partial_t \omega + u\omega_x + v\omega_y = \nu\omega_{yy} + \theta_x,$$

the estimate of any $L^q$-norm of $\omega$ is coupled with the estimate of $\nabla\theta$ in $L^q$ because of the “mismatch” between the partial derivatives of $\omega_{yy}$ and of $\theta_x$. This is exactly where the problem studied here differs from the cases previously studied. In [2] we discovered that the norm of the vertical velocity $v$ in Lebesgue space plays a crucial role in controlling the Sobolev norms of the solutions. It was shown there, among other results, that the $L^q$-norm of the vertical velocity $v$ with $2 \leqslant q < \infty$ is bounded at any time. The bound obtained in [2] depends exponentially on $q$. This paper still aims at the global regularity issue of (1.1) and we establish two major results. The first one improves the bound for $\|v\|_{L^q}$ to a linear function of $q$. More precisely, we have the following theorem. Here and in the rest of this paper $\|f\|_{L^q}$ or simply $\|f\|_q$ denotes the norm in the Lebesgue space $L^q$, and $\|f\|_{W^{s,q}}$ or simply $\|f\|_{s,q}$ denotes the norm in the Sobolev space $W^{s,q}$.

**Theorem 1.1.** *Let $2 \leqslant q < \infty$. Let $(u_0, v_0) \in L^2 \cap L^4 \cap L^q$ and $\theta_0 \in L^2 \cap L^\infty$. Let $(u, v, \theta)$ be a smooth solution of (1.1) with the initial data $(u_0, v_0, \theta_0)$. Let $T > 0$. Then, for any $0 \leqslant t \leqslant T$,*

$$\big\|v(\cdot, t)\big\|_q \leqslant C(\nu, T, u_0, v_0, \theta_0) q$$

*where $C$ depends on $\nu$, $T$ and $\|(u_0, v_0)\|_{L^2 \cap L^4 \cap L^q}$ and $\|\theta_0\|_{L^2 \cap L^\infty}$.*

The bound above in the case when $q$ is an integer is obtained by mathematical induction and in the general case by interpolation. A basic ingredient of the proof is the following global bounds on the pressure $p$,

$$\left\|p(\cdot,t)\right\|_2 \leqslant C(\nu,T,u_0,v_0,\theta_0), \qquad \int_0^t \left\|\nabla p(\cdot,\tau)\right\|_2^2 d\tau \leqslant C(\nu,T,u_0,v_0,\theta_0).$$

In order to prove these bounds on $p$, we first establish a global inequality that bounds the $L^4$-norm of $(u,v,\theta)$, namely

$$\left\|u^2+v^2\right\|_2^2 + \nu\int_0^t\int \left(u_y^2+v_y^2\right)\left(u^2+v^2\right) + \nu\int_0^t\int (uu_y+vv_y)^2 \leqslant C(\nu,T,u_0,v_0,\theta_0)$$

and then relate $p$ to $(u,v,\theta)$ through the divergence free condition.

Our second major result is the following conditional global regularity result.

**Theorem 1.2.** *Let $(u_0,v_0,\theta_0)\in H^2(\mathbb{R}^2)$ and let $(u,v,\theta)$ be the corresponding solution of (1.1). Let $T>0$. If $v$ satisfies*

$$\int_0^T \left(\sup_{2\leqslant q<\infty} \frac{\|v(\cdot,t)\|_q}{\sqrt{q}}\right)^2 dt < \infty, \tag{1.3}$$

*then $(u,v,\theta)$ remains in $H^2(\mathbb{R}^2)$ on $[0,T]$.*

In particular, if there exists a constant $C$ that may depend on $T$ and $(u_0,v_0,\theta_0)$ such that

$$\left\|v(\cdot,t)\right\|_q \leqslant C\sqrt{q},$$

then the corresponding solution $(u,v,\theta)$ remains regular on $[0,T]$. In order to prove this theorem, we combine the following interpolation inequality

$$\|f\|_\infty \leqslant \sup_{2\leqslant q<\infty}\frac{\|f\|_q}{\sqrt{q}}\left(\ln\left(1+\|f\|_{H^s}\right)\right)^{1/2}, \quad s>1 \tag{1.4}$$

with a bound that controls the $H^2$-norm of the solution by $\|v\|_{L^\infty}$, namely

$$\left\|(u,v,\theta)\right\|_{H^2}^2 + \left\|\omega^2+|\nabla\theta|^2\right\|_2^2 \leqslant C(\nu,T,u_0,v_0,\theta_0)\exp\left(\int_0^t \left\|v(\cdot,\tau)\right\|_{L^\infty}^2 d\tau\right). \tag{1.5}$$

A more general version of (1.4) is presented in Section 5. We remark that (1.5) involves the estimate of $\|(u,v,\theta)\|_{W^{1,4}}$, which serves as a bridge to the estimate in $H^2$. (1.5) is necessary for the proof of Theorem 1.2. The previous approach of controlling $H^2$-norm by $\|(u,v)\|_{H^1}$ in [2] is not good enough for this purpose.

The rest of this paper is arranged as follows. Section 2 presents the global in time $L^4$-bound. Section 3 establishes the global bounds for the pressure $p$. Section 4 proves Theorem 1.1. Section 5 proves a general version of (1.4), (1.5) and Theorem 1.2. Section 6 briefly discusses some of the potential approaches that lead to a complete resolution of the global regularity problem.

## 2. Global $L^4$-bound

This section establishes the global $L^4$-bound. This result serves as a preparation for the global bounds presented in the next two sections.

**Theorem 2.1.** *Let $(u_0, v_0, \theta_0) \in L^2 \cap L^4$. Let $(u, v, \theta)$ be a smooth solution of (1.1) emanating from $(u_0, v_0, \theta_0)$. Let $T > 0$. Then, for any $0 \leqslant t \leqslant T$,*

$$\|u^2 + v^2\|_2^2 + \nu \int_0^t \int (u_y^2 + v_y^2)(u^2 + v^2) + \nu \int_0^t \int (uu_y + vv_y)^2 \leqslant C(\nu, T, u_0, v_0, \theta_0), \quad (2.1)$$

*where $C(\nu, T, u_0, v_0, \theta_0)$ is a constant depending on $\nu$, $T$ and $\|(u_0, v_0, \theta_0)\|_{L^2 \cap L^4}$.*

We need two basic ingredients to prove this theorem and they are recalled here. The first one is a lemma that controls the integral of a triple product by the norms of the functions and of their partial derivatives. This type of inequality is very useful in the study of partial differential equations with anisotropic dissipation. The proof of this lemma can be found in [6].

**Lemma 2.2.** *Assume that $f$, $g$, $g_y$, $h$ and $h_x$ are all in $L^2(\mathbf{R}^2)$. Then,*

$$\iint |fgh|\, dx\, dy \leqslant C\|f\|_2 \|g\|_2^{1/2} \|g_y\|_2^{1/2} \|h\|_2^{1/2} \|h_x\|_2^{1/2}. \quad (2.2)$$

The second ingredient is the global $L^2$-bound for the velocity and $L^q$-bound for $\theta$. The derivation of these inequalities can be found in [2].

**Lemma 2.3.** *Let $(u, v, \theta)$ be a smooth solution of (1.1). Then*

$$\|(u(t), v(t))\|_2^2 + 2\nu \int_0^t \|(u_y(\tau), v_y(\tau))\|_2^2 \, d\tau = \left(\|(u_0, v_0)\|_2 + t\|\theta_0\|_2\right)^2 \quad (2.3)$$

*and, for any $2 \leqslant q < \infty$,*

$$\|\theta(t)\|_q^q + \kappa q(q-1) \int_0^t \left\|\theta_y |\theta|^{\frac{q-2}{2}}(\tau)\right\|_2^2 d\tau = \|\theta_0\|_q^q. \quad (2.4)$$

*In particular, for $2 \leqslant q \leqslant \infty$,*

$$\|\theta(t)\|_q \leqslant \|\theta_0\|_q. \quad (2.5)$$

**Proof of Theorem 2.1.** Let $r \geqslant 1$. Multiplying the first equation in (1.1) by $u^{2r-1}(u^{2r} + v^{2r})$ and the second equation by $v^{2r-1}(u^{2r} + v^{2r})$, integrating in space and performing integration by parts, we obtain

$$\frac{1}{4r} \int (u^{2r} + v^{2r})^2 + \nu(2r-1) \int (u_y^2 u^{2r-2} + v_y^2 v^{2r-2})(u^{2r} + v^{2r})$$
$$+ 2\nu r \int (u^{2r-1} u_y + v^{2r-1} v_y)^2 = I_1 + I_2 + I_3$$

where

$$I_1 = -\int p_x u^{2r-1}\big(u^{2r}+v^{2r}\big),$$
$$I_2 = -\int p_y v^{2r-1}\big(u^{2r}+v^{2r}\big),$$
$$I_3 = \int \theta v^{2r-1}\big(u^{2r}+v^{2r}\big).$$

We first estimate $I_1$ for the case $r=1$. By Lemma 2.2,

$$I_1 \leqslant \|p_x\|_2\|u\|_2^{1/2}\|u_x\|_2^{1/2}\big\|u^2+v^2\big\|_2^{1/2}\|2uu_y+2vv_y\|_2^{1/2}.$$

Taking the divergence of the first two equations in (1.1) leads to

$$-\Delta p = 2(vu_y)_x + 2(vv_y)_y - \theta_y. \tag{2.6}$$

By the boundedness of Riesz transforms on $L^2$,

$$\|p_x\|_2 \leqslant C\big(\|vu_y\|_2+\|vv_y\|_2+\|\theta\|_2\big).$$

Therefore, by Young's inequality and $u_x+v_y=0$,

$$I_1 \leqslant \frac{\nu}{2}\|uu_y+vv_y\|_2^2+\frac{\nu}{2}\big(\|vu_y\|_2+\|vv_y\|_2+\|\theta\|_2\big)^2+C\|u\|_2^2\|v_y\|_2^2\big\|u^2+v^2\big\|_2^2.$$

To estimate $I_2$, we first integrate by parts to obtain

$$\begin{aligned}I_2 &= (2r-1)\int pv^{2r-2}v_y\big(u^{2r}+v^{2r}\big)+2r\int pv^{2r-1}\big(u^{2r-1}u_y+v^{2r-1}v_y\big)\\ &\equiv I_{21}+I_{22}.\end{aligned}$$

By Hölder's inequality, Young's inequality and the Sobolev inequality

$$\|f\|_{4r} \leqslant Cr\|\nabla f\|_{\frac{4r}{2r+1}},$$

we have

$$\begin{aligned}I_{22} &\leqslant 2r\|p\|_{4r}\|u^{2r-1}u_y+v^{2r-1}v_y\|_2\big\|v^{2r-1}\big\|_{\frac{4r}{2r-1}}\\
&\leqslant Cr^2\|\nabla p\|_{\frac{4r}{2r+1}}\big\|u^{2r-1}u_y+v^{2r-1}v_y\big\|_2\big\|v^{2r-1}\big\|_{\frac{4r}{2r-1}}\\
&\leqslant Cr^2\big(\|vu_y+vv_y\|_{\frac{4r}{2r+1}}+\|\theta\|_{\frac{4r}{2r+1}}\big)\big\|u^{2r-1}u_y+v^{2r-1}v_y\big\|_2\|v\|_{4r}^{2r-1}\\
&\leqslant Cr^2\big(\|v\|_{4r}\big(\|u_y\|_2+\|v_y\|_2\big)+\|\theta\|_{\frac{4r}{2r+1}}\big)\big\|u^{2r-1}u_y+v^{2r-1}v_y\big\|_2\|v\|_{4r}^{2r-1}\\
&= Cr^2\big(\|v\|_{4r}^{2r}\big(\|u_y\|_2+\|v_y\|_2\big)+\|\theta\|_{\frac{4r}{2r+1}}\|v\|_{4r}^{2r-1}\big)\big\|u^{2r-1}u_y+v^{2r-1}v_y\big\|_2\\
&\leqslant \nu r\big\|u^{2r-1}u_y+v^{2r-1}v_y\big\|_2^2+Cr^3\big(\|u_y\|_2+\|v_y\|_2\big)^2\|v\|_{4r}^{4r}+\|\theta\|_{\frac{4r}{2r+1}}^2\|v\|_{4r}^{4r-2}.\end{aligned}$$

To estimate $I_{21}$, we split it into two terms

$$I_{21} = (2r-1)\int p v^{2r-2} v_y u^{2r} + (2r-1)\int p v^{2r-2} v_y v^{2r} = I_{211} + I_{212}.$$

The two terms on the right can be bounded as follows.

$$I_{211} \leqslant 2r\|p\|_{4r}\left\|u^r v^{r-1}\right\|_{\frac{4r}{2r-1}}\left\|u^r v^{r-1} v_y\right\|_2.$$

Since $\|p\|_{4r}$ can be bounded as before, we have

$$\begin{aligned} I_{211} &\leqslant Cr^2\left(\|v\|_{4r}\|u_y + v_y\|_2 + \|\theta\|_{\frac{4r}{2r+1}}\right)\|u^r\|_4\left\|v^{r-1}\right\|_{\frac{4r}{r-1}}\left\|u^r v^{r-1} v_y\right\|_2 \\ &\leqslant \frac{\nu r}{2}\left\|u^r v^{r-1} v_y\right\|_2^2 + Cr^3\|v\|_{4r}^{2r}\|u_y + v_y\|_2^2\|u\|_{4r}^{2r} + \|\theta\|_{\frac{4r}{2r+1}}^2\|u\|_{4r}^{2r}\|v\|_{4r}^{r}. \end{aligned}$$

$I_{212}$ can be similarly estimated as $I_{211}$. In fact,

$$I_{212} \leqslant \frac{\nu r}{2}\left\|v^{2r-1} v_y\right\|_2^2 + Cr^3\|v\|_{4r}^{4r}\|u_y + v_y\|_2^2 + \|\theta\|_{\frac{4r}{2r+1}}^2\|v\|_{4r}^{3r}.$$

Collecting the estimates for $I_2$, we have

$$\begin{aligned} I_2 \leqslant{}& \nu r\left\|u^{2r-1}u_y + v^{2r-1}v_y\right\|_2^2 + \frac{\nu r}{2}\left\|u^r v^{r-1} v_y\right\|_2^2 + \frac{\nu r}{2}\left\|v^{2r-1} v_y\right\|_2^2 \\ &+ Cr^3\left(\|u_y\|_2 + \|v_y\|_2\right)^2\|v\|_{4r}^{2r}\left(\|u\|_{4r}^{2r} + \|v\|_{4r}^{2r}\right) \\ &+ \|\theta\|_{\frac{4r}{2r+1}}^2\left(\|v\|_{4r}^{4r-2} + \|u\|_{4r}^{2r}\|v\|_{4r}^{r} + \|v\|_{4r}^{3r}\right). \end{aligned}$$

The estimate for $I_3$ is easy.

$$I_3 \leqslant \|\theta\|_{4r}\left\|v^{2r-1}\right\|_{\frac{4r}{2r-1}}\left\|u^{2r} + v^{2r}\right\|_2 \leqslant \|\theta\|_{4r}\left\|u^{2r} + v^{2r}\right\|_2^{2-\frac{1}{4r}}.$$

In the special case when $r = 1$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int\left(u^2 + v^2\right)^2 + \nu\int\left(u_y^2 + v_y^2\right)\left(u^2 + v^2\right) + \nu\int(uu_y + vv_y)^2 \\ &\quad\leqslant C\|u\|_2^2\|v_y\|_2^2\left\|u^2 + v^2\right\|_2^2 + C\left(\|u_y\|_2 + \|v_y\|_2\right)^2\|v\|_4^2\left(\|u\|_4^2 + \|v\|_4^2\right) \\ &\qquad + \|\theta\|_{\frac{4}{3}}^2\left(\|v\|_4^2 + \|u\|_4^2\|v\|_4 + \|v\|_4^3\right) + \|\theta\|_4\left\|u^2 + v^2\right\|_2^{2-\frac{1}{4}}. \end{aligned}$$

This inequality, together with Gronwall's inequality, yields (2.1). □

## 3. Global bounds for the pressure

The pressure can also be bounded globally.

**Theorem 3.1.** *Let $(u, v, \theta)$ be the solution as stated in Theorem 2.1. Let $p$ the associated pressure. Let $T > 0$. Then, for any $0 \leqslant t \leqslant T$,*

$$\left\|p(\cdot,t)\right\|_2 \leqslant C(\nu,T,u_0,v_0,\theta_0), \qquad \int_0^t \left\|\nabla p(\cdot,\tau)\right\|_2^2 d\tau \leqslant C(\nu,T,u_0,v_0,\theta_0), \tag{3.1}$$

*where $C(T)$ depends on $\nu$, $T$ and $\|(u_0,v_0,\theta_0)\|_{L^2\cap L^4}$.*

**Proof.** According to Theorem 2.1, $(u, v)$ obeys the $L^4$-bound

$$\int_0^t \int \left(u^2+v^2\right)\left(u_y^2+v_y^2\right) dx\, d\tau \leqslant C(\nu,T,u_0,v_0,\theta_0).$$

According to (2.6) and the boundedness of Riesz transforms on $L^2$,

$$\|\nabla p\|_2 \leqslant C\|v v_y + v u_y\|_2 + \|\theta_0\|_2.$$

Integrating in time and invoking Lemma 2.3 lead to

$$\int_0^t \left\|\nabla p(\cdot,\tau)\right\|_2^2 d\tau \leqslant C(\nu,T,u_0,v_0,\theta_0).$$

To prove the first inequality in (3.1), we have from (2.6)

$$-\Delta p = \left(u^2\right)_{xx} + (uv)_{xy} + (uv)_{yx} + \left(v^2\right)_{yy} - \theta_y.$$

Since the Riesz transforms are bounded in $L^2$, we have

$$\|p\|_2 \leqslant C\left\|(u,v)\right\|_4^2 + \left\|(-\Delta)^{-1}\partial_y\theta\right\|_2.$$

According to Theorem 2.1,

$$\left\|(u,v)\right\|_4 \leqslant C(\nu,T,u_0,v_0,\theta_0).$$

By the boundedness of Riesz transforms on $L^2$, we have

$$\left\|(-\Delta)^{-1}\partial_y\theta\right\|_2 = \left\|\Lambda^{-1}\partial_y\Lambda^{-1}\theta\right\|_2 \leqslant \left\|\Lambda^{-1}\theta\right\|_2,$$

where $\Lambda = (-\Delta)^{1/2}$. The boundedness of $\|\Lambda^{-1}\theta\|_2$ follows from a simple energy estimate. In fact, applying $\Lambda^{-1}$ to the equation for $\theta$, namely the fourth equation in (1.1) and taking the inner product with $\Lambda^{-1}\theta$, we obtain

$$\frac{d}{dt}\|\Lambda^{-1}\theta\|_2^2 + 2\kappa\|\Lambda^{-1}\theta_y\|_2^2 \leqslant \|\Lambda^{-1}(u\theta)_x + \Lambda^{-1}(v\theta)_y\|_2\|\Lambda^{-1}\theta\|_2$$
$$\leqslant \big(\|u\theta\|_2 + \|v\theta\|_2\big)\|\Lambda^{-1}\theta\|_2$$
$$\leqslant \|(u, v)\|_4\|\theta_0\|_4\|\Lambda^{-1}\theta\|_2.$$

Using the $L^4$-bound for $(u, v)$, we have

$$\|\Lambda^{-1}\theta\|_2 \leqslant \|\Lambda^{-1}\theta_0\|_2 + C(\nu, T, u_0, v_0, \theta_0).$$

Therefore,

$$\|p(\cdot, t)\|_2 \leqslant C(\nu, T, u_0, v_0, \theta_0).$$

This completes the proof of Theorem 3. □

## 4. Global $L^q$-bound for the vertical velocity

This section establishes a global $L^q$-bound for the vertical velocity. This bound is linear in $q$ and significantly improves the exponential bound of [2].

**Theorem 4.1.** *Let $2 \leqslant q < \infty$. Let $(u_0, v_0) \in L^2 \cap L^4 \cap L^q$ and $\theta_0 \in L^2 \cap L^\infty$. Let $(u, v, \theta)$ be a smooth solution of (1.1) with the initial data $(u_0, v_0, \theta_0)$. Let $T > 0$. Then, for any $0 \leqslant t \leqslant T$,*

$$\|v(\cdot, t)\|_q \leqslant C(\nu, T, u_0, v_0, \theta_0)q$$

*where $C$ depends on $\nu$, $T$ and $\|(u_0, v_0)\|_{L^2\cap L^4\cap L^q}$ and $\|\theta_0\|_{L^2\cap L^\infty}$.*

**Proof.** It suffices to prove this for positive integers $q$. The bound for a general real number $q \geqslant 2$ then follows from interpolation.

The proof for the case of positive integers $q$ is done by induction. In fact, we prove inductively that, for any $q \geqslant 2$,

$$\|v\|_q^q + \frac{\nu}{2}q(q-1)\int_0^t\int v_y^2|v|^{q-2}\,dx\,dy\,d\tau \leqslant C^q q^q \tag{4.1}$$

where $C = C(\nu, T, u_0, v_0, \theta_0)$. First of all, the bound holds for $q = 2$,

$$\|v\|_2^2 + 2\nu\int_0^t\int v_y^2\,dx\,dy\,d\tau \leqslant \|\theta_0\|_2\big(\|v_0\|_2 + t\|\theta_0\|_2\big),$$

which is a special consequence of Lemma 2.3. Similarly we can easily obtain (4.1) with $q = 3$. Multiplying the equation for $v$ in (1.1) by $v|v|$, integrating in space and applying Hölder's inequality, we have

$$\frac{d}{dt}\|v\|_3^3 + 6\nu\int v_y^2|v| \leqslant \big(\|p_y\|_2 + \|\theta_0\|_2\big)\|v\|_4^2.$$

The desired inequality then follows from the global bounds in Theorems 2.1 and 3.1. Now we make the inductive assumption, for any $k \leqslant q-1$,

$$\|v\|_k^k + \frac{\nu}{2}k(k-1)\int_0^t \int v_y^2 v^{k-2}\,dx\,dy\,d\tau \leqslant C^k k^k$$

and prove that

$$\|v\|_q^q + \frac{\nu}{2}q(q-1)\int_0^t \int v_y^2 v^{q-2}\,dx\,dy\,d\tau \leqslant C^q q^q.$$

Multiplying the equation for $v$ in (1.1) by $v|v|^{q-2}$ and integrating by parts, we obtain

$$\frac{1}{q}\frac{d}{dt}\|v\|_q^q + \nu(q-1)\int v_y^2|v|^{q-2}\,dx\,dy = I_1 + I_2,$$

where

$$I_1 = -\int p_y v|v|^{q-2}, \qquad I_2 = \int \theta v|v|^{q-2}.$$

$I_2$ is easily bounded,

$$|I_2| \leqslant \|\theta\|_\infty \|v\|_{q-1}^{q-1}.$$

Integrating by parts and applying Lemma 2.2, we have

$$\begin{aligned}
I_1 &= (q-1)\int p v_y|v|^{q-2} \\
&\leqslant (q-1)\big\|v_y|v|^{q/2-1}\big\|_2 \|p\|_2^{1/2}\|p_x\|_2^{1/2}\big\|v^{q/2-1}\big\|_2^{1/2}\big\|\big(v^{q/2-1}\big)_y\big\|_2^{1/2} \\
&\leqslant \frac{\nu}{2}(q-1)\big\|v_y v^{q/2-1}\big\|_2^2 + C(\nu)(q-2)(q-1)\|p\|_2\|p_x\|_2\|v\|_{q-2}^{\frac{q-2}{2}}\left(\int v_y^2|v|^{q-4}\right)^{1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\|v\|_q^q + \frac{(q-1)\nu}{2}\int v_y^2 v^{q-2}\,dx \\
&\quad\leqslant C(\nu)(q-2)(q-1)\|p\|_2\|p_x\|_2\|v\|_{q-2}^{\frac{q-2}{2}}\left(\int v_y^2|v|^{q-4}\right)^{1/2} + \|\theta\|_\infty\|v\|_{q-1}^{q-1}.
\end{aligned} \tag{4.2}$$

By Hölder's inequality,

$$(q-2)\|p_x\|_2\|v\|_{q-2}^{\frac{q-2}{2}}\left(\int v_y^2|v|^{q-4}\right)^{1/2} \leqslant \|p_x\|_2^2\|v\|_{q-2}^{q-2} + (q-2)^2\int v_y^2|v|^{q-4}.$$

Inserting the above inequality in (4.2), integrating in $t$, applying Theorem 3.1 and the inductive assumption, we have

$$\begin{aligned}&\|v\|_q^q+\frac{\nu}{2}(q-1)q\int_0^t\int v_y^2 v^{q-2}\\&\leqslant C^2 q(q-1)C^{q-2}(q-2)^{q-2}+q\|\theta_0\|_\infty T C^{q-1}(q-1)^{q-1}\\&\leqslant C^q q^q.\end{aligned}$$

This completes the proof of Theorem 4.1. □

## 5. A conditional global regularity

This section proves the conditional global regularity result stated in Theorem 1.2. It is restated here.

**Theorem 5.1.** *Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$ and let $(u, v, \theta)$ be the corresponding solution of (1.1). Let $T > 0$. If $v$ satisfies*

$$\int_0^T \left(\sup_{2\leqslant q<\infty}\frac{\|v(\cdot,t)\|_q}{\sqrt{q}}\right)^2 dt<\infty, \tag{5.1}$$

*then $(u, v, \theta)$ remains in $H^2(\mathbb{R}^2)$ on $[0, T]$.*

The proof of this theorem relies on two major propositions. The first one provides an interpolation inequality that bounds the $L^\infty$-norm of a function $f$ in terms of

$$\sup_{2\leqslant q<\infty}\frac{\|f\|_q}{\sqrt{q}}$$

and the logarithm of $\|f\|_{H^s}$ with $s > 1$. The second one establishes a bound for the norms of any classical solution in $W^{1,4}$ and $H^2$ in terms of $\|v\|_\infty$.

The interpolation inequality is stated and proven in the Besov space setting and the desired inequality is a special consequence. The definition of Besov space and related useful facts can be found in several books (see e.g. [4] or [8]).

**Proposition 5.2.** *Assume $f \in B^s_{r,2}(\mathbb{R}^d)$ with $2 \leqslant r \leqslant \infty$ and $s > d/r$. Let $a > 0$. If*

$$\sup_{2\leqslant q<\infty}\frac{\|f\|_q}{q^a}<\infty,$$

*then*

$$\|f\|_\infty\leqslant\sup_{2\leqslant q<\infty}\frac{\|f\|_q}{q^a}\left(\ln\left(1+\|f\|_{B^s_{r,2}}\right)\right)^a.$$

*Specifically, when $d = 2$, $a = 1/2$, $r = 2$ and $s > 1$, we have*

$$\|f\|_\infty\leqslant\sup_{2\leqslant q<\infty}\frac{\|f\|_q}{\sqrt{q}}\left(\ln\left(1+\|f\|_{H^s}\right)\right)^{1/2}.$$

**Proposition 5.3.** *Assume* $(u_0, v_0, \theta_0) \in H^2$. *Let* $(u, v, \theta)$ *be the corresponding classical solution of* (1.1)*. Then the quantity*

$$Y(t) = \|\omega\|_{H^1}^2 + \|\theta\|_{H^2}^2 + \big\|\omega^2 + |\nabla\theta|^2\big\|_2^2$$

*satisfies*

$$\begin{aligned}&\frac{d}{dt}Y(t) + \|\omega_y\|_{H^1}^2 + \|\theta_y\|_{H^2}^2 + \int \big(\omega^2 + |\nabla\theta|^2\big)\big(\omega_y^2 + |\nabla\theta_y|^2\big) + \int (\omega\omega_y + \nabla\theta\cdot\nabla\theta_y)^2\\ &\quad\leqslant C\big(1 + \|\theta_0\|_\infty^2 + \|v\|_\infty^2 + \|u_y\|_2^2 + \big(1 + \|u\|_2^2\big)\|v_y\|_2^2\big)Y(t),\end{aligned}$$

*where C is a constant.*

Theorem 5.1 follows as a consequence of the two propositions above.

**Proof of Theorem 5.1.** Integrating the inequality in Proposition 5.3 in time and employing the basic inequalities $\|\omega\|_{H^1}^2 = \|\nabla(u,v)\|_{H^1}^2$ and $\|\nabla(u,v)\|_4 \leqslant C\|\omega\|_4$, we find

$$Z(t) \equiv \big\|(u,v,\theta)(\cdot,t)\big\|_{H^2}^2 + \big\|(u,v,\theta)(\cdot,t)\big\|_{W^{1,4}}^4$$

obeys

$$Z(t) \leqslant \int_0^t \big(1 + \|\theta_0\|_\infty^2 + \|v\|_\infty^2 + \|u_y\|_2^2 + \big(1 + \|u\|_2^2\big)\|v_y\|_2^2\big)Z(\tau)\,d\tau. \tag{5.2}$$

By Proposition 5.2,

$$\|v\|_\infty^2 \leqslant \Big(\sup_{2\leqslant q<\infty}\frac{\|v\|_q}{\sqrt{q}}\Big)^2 \ln\big(1 + \|v\|_{H^2}\big) \leqslant \Big(\sup_{2\leqslant q<\infty}\frac{\|v\|_q}{\sqrt{q}}\Big)^2 \ln\big(1 + Z(t)\big). \tag{5.3}$$

Inserting (5.3) in (5.2) and applying Gronwall's inequality then lead to Theorem 5.1. □

We now prove Propositions 5.2 and 5.3.

**Proof of Proposition 5.2.** By the Littlewood–Paley decomposition, we can write

$$f = S_{N+1}f + \sum_{j=N+1}^{\infty} \Delta_j f,$$

where $\Delta_j$ denotes the Fourier localization operator and

$$S_{N+1} = \sum_{j=-1}^{N} \Delta_j.$$

The precise definition of $\Delta_j$ and $S_N$ can be found in several books and many papers (see e.g. [8]). Therefore,

$$\|f\|_\infty \leqslant \|S_{N+1}f\|_\infty + \sum_{j=N+1}^{\infty} \|\Delta_j f\|_\infty.$$

We denote the terms on the right by $I$ and $II$. By Bernstein's inequality, for any $p \geqslant 2$,

$$I \leqslant 2^{\frac{Nd}{p}} \|S_{N+1}f\|_p \leqslant 2^{\frac{Nd}{p}} \|f\|_{L^p} \leqslant 2^{\frac{Nd}{p}} p^a \sup_{q\geqslant 2} \frac{\|f\|_q}{q^a}.$$

For any $s > d/r$,

$$II \leqslant \sum_{j=N+1}^{\infty} 2^{\frac{jd}{r}} \|\Delta_j f\|_r = \sum_{j=N+1}^{\infty} 2^{j(\frac{d}{r}-s)} 2^{sj} \|\Delta_j f\|_r = C2^{(N+1)(\frac{d}{r}-s)} \|f\|_{B^s_{r,2}},$$

where $C$ is a constant depending on $s$ only. Therefore,

$$\|f\|_\infty \leqslant 2^{\frac{Nd}{p}} p^a \sup_{q\geqslant 2} \frac{\|f\|_q}{q^a} + C2^{(N+1)(\frac{d}{r}-s)} \|f\|_{B^s_{r,2}}.$$

Setting $p = N$ and $N$ to be the largest integer satisfying

$$N \leqslant \frac{1}{s-\frac{d}{r}} \log_2\left(1 + \|f\|_{B^s_{r,2}}\right),$$

the desired inequality then follows. □

The proof of Proposition 5.3 is completed in three steps. The first step bounds the $L^2$-norm of $(\omega, \nabla\theta)$, the second bounds the $L^4$-norm of $(\omega, \nabla\theta)$ while the third controls the $L^2$-norm of $(\nabla\omega, \Delta\theta)$. For the sake of clarity, we divide the whole proof into three subsections.

### *5.1. $H^1$-bound*

This subsection provides an $H^1$-bound for classical solutions of (1.1) in terms of $\|v\|_\infty$. This bound is essentially Proposition 3.3 in [2].

**Proposition 5.4.** *Let $(u, v, \theta)$ be a classical solution of* (1.1) *emanating from the initial data $(u_0, v_0, \theta_0)$. Then*

$$\frac{d}{dt}\left(\|\omega\|_2^2 + \|\nabla\theta\|_2^2\right) + \|\omega_y\|_2^2 + \|\nabla\theta_y\|_2^2$$
$$\leqslant C\left(1 + \|\theta_0\|_\infty^2 + \|v\|_\infty^2 + \|u_y\|_2^2 + \|v_y\|_2^2\right)\left(\|\omega\|_2^2 + \|\nabla\theta\|_2^2\right).$$

**Proof.** The proof is similar to the one for Proposition 3.3 in [2]. The only difference is that we estimate $\|\omega\|_2$ here instead of $\|(\nabla u, \nabla v)\|_2$. We omit further details. □

### 5.2. $W^{1,4}$-bound

This subsection presents a proposition that bounds the $W^{1,4}$-norm of classical solutions to (1.1) in terms of $\|v\|_\infty$.

**Proposition 5.5.** *Let $(u, v, \theta)$ be a classical solution of (1.1) corresponding to the initial data $(u_0, v_0, \theta_0)$. Then*

$$\frac{d}{dt}\int E^2 + \int E\big(\omega_y^2 + |\nabla\theta_y|^2\big) + \int (\omega\omega_y + \nabla\theta\cdot\nabla\theta_y)^2 \leqslant C\big(\|v\|_\infty^2 + \|u\|_2^2\|v_y\|_2^2 + \|\theta_0\|_\infty^2\big)\int E^2,$$

*where*

$$E = \omega^2 + |\nabla\theta|^2.$$

**Proof.** We estimate the $L^4$-norm of $(\omega, \nabla\theta)$. For the sake of simplicity, we set $\nu = \kappa = 1$. It is clear from (1.1) that $\omega \equiv v_x - u_y$ satisfies

$$\partial_t\omega + u\omega_x + v\omega_y = \omega_{yy} + \theta_x. \tag{5.4}$$

Multiplying (5.4) by $\omega E$, dotting $\nabla$ of the equation for $\theta$ by $\nabla\theta E$, integrating on $\mathbb{R}^2$ and applying the divergence-free condition $u_x + v_y = 0$, we obtain after integration by parts,

$$\frac{1}{4}\frac{d}{dt}\int E^2 + \int E\big(\omega_y^2 + |\nabla\theta_y|^2\big) + 2\int (\omega\omega_y + \nabla\theta\cdot\nabla\theta_y)^2 = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \int \theta_x\omega E, \qquad I_2 = -\int u_x\theta_x^2 E, \qquad I_3 = -\int u_y\theta_x\theta_y E,$$
$$I_4 = -\int v_x\theta_x\theta_y E, \qquad I_5 = -\int v_y\theta_y^2 E.$$

These terms can be bounded as follows. Clearly,

$$|I_1| \leqslant \frac{1}{2}\int E^2.$$

By the divergence-free condition $u_x + v_y = 0$ and integration by parts,

$$I_2 = \int v_y\theta_x^2 E = -2\int v\theta_x\theta_{xy}E - 2\int v\theta_x^2(\omega\omega_y + \nabla\theta\cdot\nabla\theta_y).$$

Therefore, by Young's inequality,

$$|I_2| \leqslant \frac{1}{4}\int \theta_{xy}^2 E + 4\int v^2\theta_x^2 E + \frac{1}{4}\int (\omega\omega_y + \nabla\theta\cdot\nabla\theta_y)^2 + 4\int v^2\theta_x^4$$
$$\leqslant \frac{1}{4}\int \theta_{xy}^2 E + \frac{1}{4}\int (\omega\omega_y + \nabla\theta\cdot\nabla\theta_y)^2 + 8\|v\|_\infty^2\int E^2.$$

$I_5$ can be bounded similarly.

$$|I_5| \leqslant \frac{1}{4}\int \theta_{yy}^2 E + \frac{1}{4}\int (\omega\omega_y + \nabla\theta\cdot\nabla\theta_y)^2 + 8\|v\|_\infty^2 \int E^2.$$

The estimates for $I_3$ and $I_4$ are more complex. To estimate $I_3$, we first integrate by parts to obtain

$$\begin{aligned} I_3 &= \int u\theta_{xy}\theta_y E + \int u\theta_x\theta_{yy}E + 2\int u\theta_x\theta_y(\omega\omega_y + \nabla\theta\cdot\nabla\theta_y) \\ &= I_{31} + I_{32} + I_{33}. \end{aligned}$$

By Lemma 2.2,

$$I_{31} = \int u\theta_{xy}E^{1/2}\theta_y E^{1/2} \leqslant C\left\|\theta_{xy}E^{1/2}\right\|_2 \|u\|_2^{1/2}\|u_x\|_2^{1/2}\left\|\theta_y E^{1/2}\right\|_2^{1/2}\left\|\left(\theta_y E^{1/2}\right)_y\right\|_2^{1/2}. \tag{5.5}$$

Since $|\theta_y E^{-1/2}| \leqslant 1$,

$$\begin{aligned} \left\|\left(\theta_y E^{1/2}\right)_y\right\|_2 &= \left\|\theta_{yy}E^{1/2} + \theta_y E^{-1/2}(\omega\omega_y + \nabla\theta\cdot\nabla\theta_y)\right\|_2 \\ &\leqslant \left\|\theta_{yy}E^{1/2}\right\|_2 + \left\|(\omega\omega_y + \nabla\theta\cdot\nabla\theta_y)\right\|_2. \end{aligned} \tag{5.6}$$

Inserting (5.6) in (5.5) and applying Young's inequality, we have

$$|I_{31}| \leqslant \frac{1}{16}\left\||\nabla\theta_y|E^{1/2}\right\|_2^2 + \frac{1}{16}\|\omega\omega_y + \nabla\theta\cdot\nabla\theta_y\|_2^2 + C\|u\|_2^2\|v_y\|_2^2\left\|\theta_y E^{1/2}\right\|_2^2.$$

The estimate for $I_{32}$ is similar.

$$|I_{32}| \leqslant \frac{1}{16}\left\||\nabla\theta_y|E^{1/2}\right\|_2^2 + \frac{1}{16}\|\omega\omega_y + \nabla\theta\cdot\nabla\theta_y\|_2^2 + C\|u\|_2^2\|v_y\|_2^2\left\|\theta_x E^{1/2}\right\|_2^2.$$

To bound $I_{33}$, we apply Lemma 2.2 again to obtain

$$\begin{aligned} I_{33} &= 2\int u\theta_x\theta_y(\omega\omega_y + \nabla\theta\cdot\nabla\theta_y) \\ &\leqslant C\|\omega\omega_y + \nabla\theta\cdot\nabla\theta_y\|_2\|u\|_2^{1/2}\|u_x\|_2^{1/2}\|\theta_x\theta_y\|_2^{1/2}\left\|(\theta_x\theta_y)_y\right\|_2^{1/2} \\ &\leqslant \frac{1}{16}\|\omega\omega_y + \nabla\theta\cdot\nabla\theta_y\|_2^2 + \frac{1}{16}\|\theta_{xy}\theta_y + \theta_{yy}\theta_x\|_2^2 + C\|u\|_2^2\|v_y\|_2^2\|\theta_x\theta_y\|_2^2 \\ &\leqslant \frac{1}{16}\|\omega\omega_y + \nabla\theta\cdot\nabla\theta_y\|_2^2 + \frac{1}{16}\left\||\nabla\theta||\nabla\theta_y|\right\|_2^2 + C\|u\|_2^2\|v_y\|_2^2\int E^2. \end{aligned}$$

Substituting $v_x = u_y + \omega$ in $I_4$, we have

$$I_4 = -\int \omega\theta_x\theta_y E - I_3.$$

Integration by parts yields

$$\int \omega\theta_x\theta_y E = -\int \omega_y\theta_x\theta E - \int \omega\theta_{xy}\theta E - 2\int \omega\theta_x\theta(\omega\omega_y + \nabla\theta\cdot\nabla\theta_y).$$

Thus

$$\begin{aligned}\left|\int \omega\theta_x\theta_y E\right| &\leqslant \frac{1}{16}\int \omega_y^2 E + C\|\theta\|_\infty^2\int \theta_x^2 E + \frac{1}{16}\int \theta_{xy}^2 E + C\|\theta\|_\infty^2\int \omega^2 E\\ &\quad + \frac{1}{16}\|\omega\omega_y + \nabla\theta\cdot\nabla\theta_y\|_2^2 + C\|\theta\|_\infty^2\int \omega^2\theta_x^2\\ &\leqslant \frac{1}{16}\int E\big(\omega_y^2 + |\nabla\theta_y|^2\big) + \frac{1}{16}\|\omega\omega_y + \nabla\theta\cdot\nabla\theta_y\|_2^2 + C\|\theta\|_\infty^2\int E^2.\end{aligned}$$

Collecting all the estimates, we obtain

$$\begin{aligned}&\frac{d}{dt}\int E^2 + \int E\big(\omega_y^2 + |\nabla\theta_y|^2\big) + \int(\omega\omega_y + \nabla\theta\cdot\nabla\theta_y)^2\\ &\quad\leqslant C\big(\|v\|_\infty^2 + \|u\|_2^2\|v_y\|_2^2 + \|\theta_0\|_\infty^2\big)\int E^2.\end{aligned}$$

This completes the proof of Proposition 5.5. □

### 5.3. $H^2$-bound

This subsection presents an *a priori* estimate for the $H^2$-norm of classical solutions to (1.1). When combined with the $W^{1,4}$ estimate in the previous subsection, this bound would result in a global bound in terms of $\|v\|_\infty$.

**Proposition 5.6.** *If $(u, v, \theta)$ is a classical solution of (1.1) emanating from $(u_0, v_0, \theta_0)$. Then*

$$\begin{aligned}&\frac{d}{dt}\big(\|\nabla\omega\|_2^2 + \|\Delta\theta\|_2^2\big) + \|\nabla\omega_y\|_2^2 + \|\Delta\theta_y\|_2^2\\ &\quad\leqslant \frac{1}{8}\big\||\nabla\theta||\nabla\theta_y|\big\|_2^2 + C\|E\|_2^2 + C\big(1 + \|\theta_0\|_\infty^2 + \|v\|_\infty^2 + \|u_y\|_2^2 + \|v_y\|_2^2\big)\big(\|\nabla\omega\|_2^2 + \|\Delta\theta\|_2^2\big).\end{aligned}$$

**Proof.** Dotting $\nabla$ of (5.4) with $\nabla\omega$, multiplying $\Delta$ of the equation for $\theta$ in (1.1) by $\Delta\theta$, integrating over $\mathbb{R}^2$ and applying the divergence-free condition $u_x + v_y = 0$, we obtain after integration by parts

$$\frac{1}{2}\frac{d}{dt}\int\big(|\nabla\omega|^2 + (\Delta\theta)^2\big) + \|\nabla\omega_y\|_2^2 + \|\Delta\theta_y\|_2^2 = J_1 + \cdots + J_9,$$

where

$$J_1 = \int \nabla\theta_x\cdot\nabla\omega,\qquad J_2 = -\int u_x\omega_x^2,\qquad J_3 = -\int u_y\omega_x\omega_y,$$

$$J_4 = -\int v_x\omega_x\omega_y,\qquad J_5 = -\int v_y\omega_y^2,\qquad J_6 = -\int \Delta u\theta_x\Delta\theta,$$

$$J_7 = -\int \Delta v\theta_y\Delta\theta,\qquad J_8 = -2\int \nabla u\cdot\nabla\theta_x\Delta\theta,\qquad J_9 = -2\int \nabla v\cdot\theta_y\Delta\theta.$$

The rest of the proof is devoted to bounding these terms.

$$|J_1| \leqslant \|\nabla\theta_x\|_2\|\nabla\omega\|_2 \leqslant \frac{1}{2}\big(\|\Delta\theta\|_2^2 + \|\nabla\omega\|_2^2\big).$$

Using $u_x + v_y = 0$ and integrating by parts, we obtain

$$J_2 = -\int v\omega_x\omega_{xy} \leqslant \frac{1}{16}\|\omega_{xy}\|_2^2 + C\|v\|_\infty^2\|\omega_x\|_2^2.$$

By Lemma 2.2,

$$|J_3| \leqslant C\|u_y\|_2\|w_x\|_2^{1/2}\|w_{xy}\|_2^{1/2}\|w_y\|_2^{1/2}\|w_{xy}\|_2^{1/2} \leqslant \frac{1}{16}\|\omega_{xy}\|_2^2 + C\|u_y\|_2^2\|\nabla\omega\|_2^2.$$

By substituting $v_x = \omega + u_y$ in $J_4$, we have

$$J_4 = J_3 - \int \omega\omega_x\omega_y.$$

Integration by parts yields

$$\left|\int \omega\omega_x\omega_y\right| \leqslant \frac{1}{16}\|\omega_{xy}\|_2^2 + C\|\omega\|_4^4.$$

$J_5$ can be similarly bounded as $J_2$.

$$|J_5| \leqslant \frac{1}{16}\|\nabla\omega_y\|_2^2 + C\|v_y\|_2^2\|w_y\|_2^2.$$

To deal with $J_6$, we split it into two terms. Integration by parts yields

$$\begin{aligned} J_6 &= -\int (\Delta u\theta_x\theta_{xx} + \Delta u\theta_x\theta_{yy}) = \frac{1}{2}\int \Delta u_x\theta_x^2 - \int \Delta u\theta_x\theta_{yy} \\ &= -\frac{1}{2}\int \Delta v_y\theta_x^2 - \int \Delta u\theta_x\theta_{yy} = \int \Delta v\theta_x\theta_{xy} - \int \Delta u\theta_x\theta_{yy}. \end{aligned}$$

Therefore,

$$|J_6| \leqslant \big\|\Delta(u,v)\big\|_2\|\theta_x\nabla\theta_y\|_2 \leqslant \frac{1}{16}\|\theta_x\nabla\theta_y\|_2^2 + C\|\nabla\omega\|_2^2.$$

By integration by parts,

$$\begin{aligned} J_7 &= -\int (v_{xx}\theta_y\theta_{xx} + v_{yy}\theta_y\theta_{xx} + \Delta v\theta_y\theta_{yy}) \\ &= \int (v_{xxy}\theta\theta_{xx} + v_{xx}\theta\theta_{xxy} + v_{yyy}\theta\theta_{xx} + v_{yy}\theta\theta_{xxy} + \Delta v\theta_y\theta_{yy}). \end{aligned}$$

Thus,

$$|J_7| \leqslant \frac{1}{16}\|\nabla\omega_y\|_2^2 + \frac{1}{16}\|\Delta\theta_y\|_2^2 + \frac{1}{16}\|\theta_y\nabla\theta_y\|_2^2 + C\big(1 + \|\theta_0\|_\infty^2\big)\big(\|\nabla\omega\|_2^2 + \|\Delta\theta\|_2^2\big).$$

For $J_8$, we use $u_x + v_y = 0$ and integrate by parts to obtain

$$J_8 = -2\int (u_x\theta_{xx}\Delta\theta + u_y\theta_{xy}\Delta\theta) = -2\int (v\theta_{xxy}\Delta\theta + v\theta_{xx}\Delta\theta_y + u_y\theta_{xy}\Delta\theta).$$

By Hölder's inequality and Lemma 2.2,

$$\begin{aligned}|J_8| &\leqslant \frac{1}{32}\|\Delta\theta_y\|_2^2 + C\|v\|_\infty^2\|\Delta\theta\|_2^2 + C\|u_y\|_2\|\theta_{xy}\|_2^{1/2}\|\theta_{xyx}\|_2^{1/2}\|\Delta\theta\|_2^{1/2}\|\Delta\theta_y\|_2^{1/2}\\ &\leqslant \frac{1}{16}\|\Delta\theta_y\|_2^2 + C\big(\|v\|_\infty^2 + \|u_y\|_2^2\big)\|\Delta\theta\|_2^2.\end{aligned}$$

To estimate $J_9$, we need to redistribute the derivatives. By integration by parts,

$$\begin{aligned}J_9 &= -2\int \nabla v\cdot\nabla\theta_y(\theta_{xx}+\theta_{yy})\\ &= 2\int (\nabla v_x\cdot\nabla\theta_y\theta_x + \nabla v\cdot\nabla\theta_{xy}\theta_x + \nabla v_y\cdot\nabla\theta_y\theta_y + \nabla v\cdot\nabla\theta_{yy}\theta_y).\end{aligned}$$

Therefore,

$$|J_9| \leqslant \frac{1}{16}\big\||\nabla\theta||\nabla\theta_y|\big\|_2^2 + \frac{1}{16}\|\Delta\theta_y\|_2^2 + C\|\nabla\omega\|_2^2 + C\|\omega\|_4^2\|\nabla\theta\|_4^2.$$

Connecting all the estimates, we obtain

$$\begin{aligned}&\frac{d}{dt}\big(\|\nabla\omega\|_2^2 + \|\Delta\theta\|_2^2\big) + \|\nabla\omega_y\|_2^2 + \|\Delta\theta_y\|_2^2\\ &\quad\leqslant \frac{1}{8}\big\||\nabla\theta||\nabla\theta_y|\big\|_2^2 + C\|E\|_2^2 + C\big(1+\|\theta_0\|_\infty^2 + \|v\|_\infty^2 + \|u_y\|_2^2 + \|v_y\|_2^2\big)\big(\|\nabla\omega\|_2^2 + \|\Delta\theta\|_2^2\big).\end{aligned}$$

This completes the proof of Proposition 5.6. □

## 6. Conclusion and discussion

We have investigated the global regularity issue concerning solutions of the 2D Boussinesq equations with vertical dissipation and vertical thermal diffusion. We established that the vertical velocity $v$ of any classical solution $(u, v, \theta)$ must satisfy, for any $T > 0$,

$$\big\|v(\cdot,t)\big\|_q \leqslant Cq, \quad 2\leqslant q<\infty,\ 0\leqslant t\leqslant T$$

for some constant $C$ depending on $T, \nu, u_0, v_0$ and $\theta_0$ only. On the other hand, if the condition

$$\int_0^T \sup_{2\leqslant q<\infty}\frac{\|v(\cdot,t)\|_q^2}{q}\,dt < \infty$$

is fulfilled, then the corresponding solution preserves its smoothness on $[0, T]$. In particular,

$$\big\|v(\cdot,t)\big\|_q \leqslant C\sqrt{q} \tag{6.1}$$

for some $C$ independent of $q$ is sufficient for the global regularity. It remains open if (6.1) is indeed satisfied. Some preliminary numerical results computed for (1.1) involving several initial data indicate that (6.1) is actually true [3].

We have made attempts to verify (6.1). One direction would be to verify that the pressure $p$ satisfies the global bound, for any $T > 0$,

$$\int_0^t \|p\|_{2r}^2 \, dt \leqslant C(T), \quad t \leqslant T \tag{6.2}$$

for some constant $C$ independent of $r$. It is not very hard to check that

$$\frac{\|v(\cdot, t)\|_{2r}^2}{2r-1} \leqslant e^{2\|\theta_0\|_{2r} t} \left( \frac{\|v_0\|_{2r}^2}{2r-1} + C \int_0^t \|p\|_{2r}^2 \, dt \right)$$

and (6.1) is an immediate consequence of (6.2). The global bounds on $p$ stated in Theorem 3.1 in Section 3 is not sufficient to prove (6.2). The Sobolev embedding inequality

$$\|p\|_{2r} \leqslant Cr \|p\|_2^{\frac{1}{r}} \|\nabla p\|_2^{1-\frac{1}{r}}$$

generates a coefficient that depends linearly on $r$. Some potential approaches to prove (6.1) include the use of the duality of the BMO space and the Hardy space $\mathcal{H}^1$. The motivation behind this is to combine the inequality $\|p\|_{BMO} \leqslant C\|p\|_{H^1}$ and the global bound $\int_0^T \|p\|_{H^1}^2 \, dt < \infty$.

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