2007 OSU HIGH SCHOOL MATH CONTEST

PART II

1. A *graph* consists of a finite set V of vertices and a set E of edges joining different pairs of distinct vertices. The number of edges incident with a vertex is called the *degree* of the vertex.

   For example, Figure 1 is a picture of a graph with four vertices A, B, C, and D and four edges (note that the point where the diagonals of the square intersect is not a vertex; the crossing edges are just an artifact of the pictorial representation of the graph.) Vertex A has degree 3, vertices B and D each have degree 2, and vertex C has degree 1.

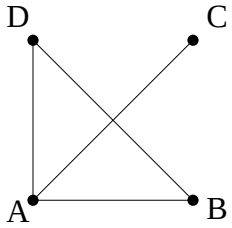

FIGURE 1

   (i) Show that the sum of the degrees of all vertices is equal to twice the number of edges in any graph.

   (ii) Show that the number of vertices of odd degree is even in any graph.

2. (i) Prove that $\sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}} = 1+\frac{1}{n(n+1)}$.
   (ii) Compute

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{2006^2}+\frac{1}{2007^2}}.$$

3. For n an integer greater than or equal to 2, prove that the sum $1 + 2 + 3 + \cdots + n$ of first n integers is a factor of the product $1 \cdot 2 \cdot 3 \cdots n$ if and only if $n + 1$ is not prime.

4. Let C, D be points on the line segment AB such that $AC \cong CD \cong DB$. AC, CD and DB are diameters of the 3 circles shown in Figure 2. Let E be any point on the center circle other than C and D. Let α denote the angle $\angle AEC$ and let the β denote the angle $\angle BED$. Show that

$$4(\tan\alpha)\cdot(\tan\beta) = 1.$$

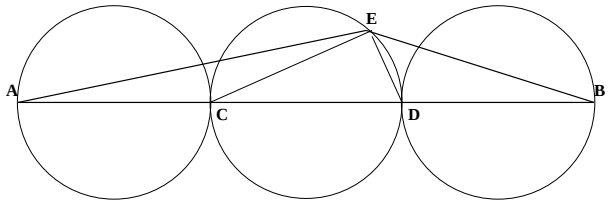

FIGURE 2

5. Let ABC be an isosceles triangle.

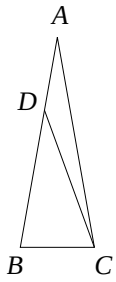

FIGURE 3

Given $AB \cong AC$ and $\angle BAC = 20°$. Let D be a point on the side AB with $AD \cong BC$.

Find the angle $\angle BDC$. You must provide a correct mathematical argument to justify your answer.

Suggestion: Consider the following figure with $\Delta ABC \cong \Delta EAD$. You may find it helpful to make further additions to this figure.

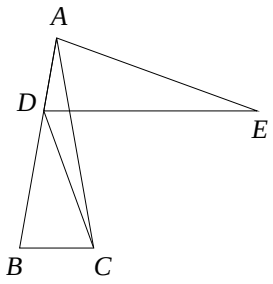

FIGURE 4