Master's Comprehensive Exam - NUMERICAL ANALYSIS - Aug. 1989

INSTRUCTIONS: You may use a calculator. Show eight digits in all numerical values that you are required to compute.

1. For a certain computer with 54–bit binary arithmetic the floating point conversion operator fl satisfies

$$fl(a) = a(1+\delta_a),\ |\delta_a| \le \epsilon,\ \epsilon = 2^{-54}.$$

Given floating point numbers x, y, z define $E = xy + z$, an exact mathematical expression. Define F to be the result of computing E in floating point arithmetic.

a) State an expression for F in terms of the data x, y, z and the floating point operator fl.

b) Prove the rounding error may be estimated by

$$|E - F| \le 1.01\epsilon(2|xy| + |z|).$$

2. Determine the Hermite cubic polynomial $p_3(x)$ for $f(x) = \sin x$ at the two points $x = 0,\ \frac{\pi}{2}$, *i.e.*, the cubic polynomial $p_3(x)$ is to satisfy the conditions $p_3(x) = f(x)$ and $p_3'(x) = f'(x)$ for $x = 0$ and $x = \frac{\pi}{2}$.

3. a) State a definition of the composite Simpson rule S_{2n} for approximating $\int_a^b f(x)dx$ based on evaluation of f(x) at the $2n+1$ equally spaced nodes $x_k,\ k = 0, \ldots, 2n$, where $x_0 = a,\ x_{2n} = b$.

b) State an estimate for the error $E_{2n}^S = |\int_a^b f(x)dx - S_{2n}|$ in terms of the stepsize $h = x_k - x_{k-1}$. [Choose from several different theorems, all involving derivatives of $f(x)$.]

c) Apply the above estimate of E_{2n}^S to $\int_0^\pi \sin x dx$ when $n = 100$.

4. Let the matrix $A = \begin{pmatrix} 1 & .99 \\ .98 & .97 \end{pmatrix}$. Solving $Ax = b$ with $b = \begin{pmatrix} 1.99 \\ 1.95 \end{pmatrix}$ we obtain the solution vector $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, while the right–hand side vector $b = \begin{pmatrix} 1.98 \\ 1.96 \end{pmatrix}$ gives a very different result.

a) Explain the dicrepancy between the two solution vectors by means of a suitable inequality that involves a condition number K of A.

b) How large is the condition number K for this matrix? Is its magnitude sufficient to explain the above discrepancy? Your answer should be in quantitative terms.