MASTER'S COMPREHENSIVE EXAM–*NUMERICAL ANALYSIS*
AUGUST 1992

Instructions: Show all work in detail and numerical values to eight (8) figures at least, or exactly. Calculator required (minimum display eight (8) figures).

1. In an equally spaced table of a function $f(x)$, with stepsize $\Delta x_k = h$, **cubic interpolation in the central interval** is to be applied at four succesive tabular points (x_i, y_i), with $\leq x_i = x_0 + ih$, $i = 0, 1, 2, 3$. This means that we obtain a cubic interpolating polynomial $p_3(x)$, that is to be evaluated for $x_1 < x < x_2$ only, hence the reference to the *central interval*. For the special case of $f(x) = x^4$ **determine exactly the largest possible error** $|e_3(x)| = |f(x) - p_3(x)|$ **when** x **lies in the central interval**.

HINT: Begin by stating an appropriate formula for the error term. Your result must be a function of the stepsize h.

2. To within 8 figure accuracy at least, compute a real root of the equation

$$f(x) = x^5 - 4x^4 + 9x^3 - 11x^2 + 6x - 2 = 0$$

by any method.

3. Data (x_i, y_i), $i = 0, 1, \ldots, 5$, tabulated below (at right) are equally spaced abscissae and corresponding y-values of a certain function $y = f(x)$.

a) Compute the table of first and second differences Δy_i, $\Delta^2 y_i$, $i = 0, \ldots, 4$ (resp. 3) .

b) From the 2nd differences that you have computed, **prove the inequality**

$$0.021 \leq \max_{1 \leq x \leq 2} |f''(x)|.$$

Here, $f''(x)$ is the 2nd derivative of the function $y = f(x)$, that is tabulated at right.

x	$y = f(x)$
0.0	0.23607
0.2	0.28035
0.4	0.32379
0.6	0.36643
0.8	0.40832
1.0	0.44949

4. The three-point **open Newton-Cotes** quadrature formula is given by the equation

$$\int_{-1}^{1} f(x)\,dx = A_1 f(-\frac{1}{2}) + A_2 f(0) + A_3 f(\frac{1}{2}) + E_3^{NC^{op}}.$$

The error term is known to satisfy $E_3^{NC^{op}} = c_4 f^{(4)}(\tilde{x})$, $\quad -1 < \tilde{x} < 1$.

a) Determine the weights A_k, $k = 1, 2, 3$.

HINT: Substitute for $f(x)$ several functions such that $f^{(4)}(x) \equiv 0$, to obtain three equations in three unknowns, noting that

$$f^{(4)}(x) \equiv 0 \Rightarrow E_3^{NC^{op}} = 0.$$

b) Compute the error constant c_4 from the fact that

$$f(x) = x^4 \Rightarrow f^{(4)}(x) \equiv 24.$$