MASTER'S COMPREHENSIVE EXAM–*NUMERICAL ANALYSIS*

AUGUST 1993

INSTRUCTION: Calculator required for some problems. Show at least eight (8) figures in numerical results, or exact answer if available.

1. For a binary floating point machine with t-bit mantissa, $t = 24$, let fl(x) denote the symmetrically rounded machine approximation of x, $x \in \mathbf{R}$ (barring over- or underflow). Let $x = 2^{-1} + 2^{-24}$, $y = 2^{-2} + 2^{-25}$, $z = 2^{-3} + 2^{-26}$.

a) Are these machine numbers? Given machine numbers x, y explain on what grounds one would claim machine addition can be defined by the mathematical operation $x \hat{+} y = \mathrm{fl}(x + y)$ (again, barring over- or underflow).

b) Compute $s_1 = (x \hat{+} y) \hat{+} z$, $s_2 = x \hat{+} (y \hat{+} z)$.

c) Is machine addition associative in this example?

2. This problem considers the (2n-1)-st degree Lagrange interpolation polynomial $p_n(x)$ at equidistant nodes x_k, $k \in \{1, 2, \cdots, n-1, n, n+1, n+2, \cdots, 2n-1, 2n\}$, given by

$$x_k = \frac{2k - 2n - 1}{2n - 1} \in [-1, 1], k = 1, \ldots, 2n.$$

Equivalently, the interpolation nodes (in a different order) are given by

$$x_k = \pm\frac{1}{2n-1},\ \pm\frac{3}{2n-1}, \cdots, \pm\frac{2n-3}{2n-1},\ \pm\frac{2n-1}{2n-1} = \pm 1.$$

a) Using $M_{2n} = \dfrac{1}{(2n)!} \max\limits_{-1 \le x \le 1} |f^{(2n)}(x)| < \infty$, state a general upper bound estimate for the interpolation error

$$e_{2n-1}(\bar{x}) = |f(\bar{x}) - p_{2n-1}(\bar{x})|,$$

for a function $f(x)$, defined and $(2n)$–times continuously differentiable on the interval $[-1, 1]$.

b) For the *special nodes* stated above and for the *special interval* $\dfrac{-1}{2n-1} < x < \dfrac{1}{2n-1}$ show $e_{2n-1}(\bar{x}) \le M_{2n} \prod\limits_{k=1}^{2n} x_k^2$.

M O R E O N B A C K ...

3. By Newton's method compute a real root x of the equation

$$f(x) = x^5 - 4x^4 + 9x^3 - 8x^2 + 5x - 1 = 0,$$

beginning with $x_0 = 0.5$. Be careful to state as many accurate iterates as possible.

4. Solve the system $A\mathbf{x} = \mathbf{b}$, where $A = QR$, with $A,\ Q,\ R, \mathbf{b}$ given below.

a) Show that $Q^T Q = I$ (the 4×4 identity matrix).

b) Letting $\mathbf{y} = R\mathbf{x}$ obtain an algorithm to solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ which requires only minimal computational effort in its execution.

HINT: The desired algorithm involves two easily solved linear systems. Don't compute A^{-1} explicitely!

c) Solve the given system by means of the algorithm that you have devised in part **b)** above.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 6 \end{bmatrix}, \quad Q = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$R = \begin{bmatrix} 2 & 3 & 4 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

5. The four-point *Gauss-Lobatto* quadrature formula is given by the equation

$$\int_{-1}^{1} f(x)\,dx = A_1 f(-1) + A_2 f(1) + A_3 f(\sqrt{0.2}) + A_4 f(-\sqrt{0.2}) + E_6^{Lob}.$$

The error term is known to satisfy $E_6^{Lob} = c_6 f^{(6)}(\tilde{x}), \quad -1 < \tilde{x} < 1.$

a) Determine the weights $A_k,\ k = 1, 2, 3, 4.$

b) Compute the error constant c_6.

HINT: Substitute $f(x) = x^m,\ m = 0, 1, \ldots, 6.$