PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—August 1997

Notation: $B = \{z \in \mathbf{C}: |z| < 1\}$, the open unit disc

1. Let f be a function defined on an open set G in $\mathbf{C}$.

   a) Define in terms of difference quotients what it means for f to be analytic in G.

   b) Give an equivalent definition of the analyticity of f in terms of power series.

   c) In this part we take $G = B$. Suppose that ϕ is a continuous function on ∂B and define f on B by

   $$f(z) = \int_\gamma \frac{\phi(w)}{w - z}\, dw\ ,$$

   where $\gamma(t) = e^{it}$ on $[0, 2\pi]$. Use the definition in part b) to show that f is analytic in B.

2. a) Show that the real and imaginary parts of an analytic function are harmonic.

   b) If u and v are real-valued harmonic functions, does it follow that $u + iv$ is analytic? Explain your answer.

3. Fix a complex number λ so that $\lambda^3 = 1$ but $\lambda \neq 1$. Let f be an entire function so that $f(\lambda z) = f(z)$ for all $z \in \mathbf{C}$. Prove using series that there exists an entire function g so that $f(z) = g(z^3)$ for all $z \in \mathbf{C}$.

4. Suppose f is analytic on a neighborhood of $\overline{B}$ and $|f(z)| \leq 1$ when $|z| = 1$. In parts a) and b) give the best upper bound you can for the given quantity.

   a) $|f(\frac{1}{2})|$

   b) $|f'(\frac{1}{2})|$

   c) Do part a) again, but make the additional assumption that $f(0) = 0$.

5. Let $G = \{x + iy: x \in \mathbf{R},\ y > x^2\}$.

   a) Show that for each $n \in \mathbf{N}$ there exists a one-to-one analytic map f_n of G onto B so that $f_n(i) = 1 - \frac{1}{n}$.

   b) Does it follow that a sequence $\{f_n\}$ as in part a) has a subsequence converging normally on G (i.e., uniformly on compact subsets of G)? If it does, what can you say about the limit function of such a subsequence? If it does not, explain why not.

6. a) State the Weierstrass theorem characterizing the zeros of entire functions.

   b) Show that every meromorphic function on $\mathbf{C}$ can be written as the quotient of two entire functions.