MASTER'S COMPREHENSIVE EXAMS *NUMERICAL ANALYSIS*

AUGUST 1999

INSTRUCTIONS: Calculator required. Show eight or more decimal figures. Two problems completely solved will be sufficient for a passing grade.

1. For a certain computer with 54–bit binary arithmetic the floating point conversion operator **fl** satisfies

$$\mathbf{fl}(a) = a(1+\delta_a),\ |\delta_a| \le \epsilon,\ \epsilon = 2^{-54}.$$

Given a positive integer n and floating point numbers $x_j,\ y_j,\ j = 1, \ldots, n$, define $z = x \bullet y = \sum_{j=1}^{n} x_j y_j$, where sum and product refer to exact mathematical arithmetic. Define $\hat{z} = x \hat{\bullet} y$ to be the result of computing $x \bullet y$ in 54–bit floating point arithmetic.

a) State an explicit recursive definition for $\hat{z}$ in terms of the data $x_j,\ y_j$ and the floating point operator **fl**.

b) For the special case of $n = 3$, prove the rounding error $|\hat{z} - z|$ may be estimated by

$$|\hat{z} - z| \le (3.1)\,\epsilon \sum_{j=1}^{3} |x_j|\,|y_j|.$$

2. Given a function $f(x) \in \mathbf{C}^{(6)}([a,b])$, denote by $p_5(x)$ the Lagrange interpolation polynomial of degree at most five such that $p_5(x_j) = f(x_j)$ at equally spaced points $x_j = a + j\,h$ with stepsize $h = \frac{b-a}{5}$, $j = 0, 1, \ldots, 5$. Also, let $M_6 = \max_{a \le x \le b} |f^{(6)}(x)|$. We seek an upper bound for the interpolation error in the "central interval" that should have the form

$$(*) \qquad |f(x) - p_5(x)| \ \le\ K\,M_6\,h^6, \quad x_2 < x < x_3.$$

Determine the constant K as small as you can so that $(*)$ is valid.

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3. Use Newton's method to solve the system of nonlinear equations:

$$1 + x - y^2 = 0$$
$$y - x^3 = 0$$

a) Start with $(x, y) = (1.5, 1.5)$, and continue until

$$||(x_{n+1}, y_{n+1}) - (x_n, y_n)||_\infty < 0.001.$$

b) Explain why Newton's method converges in this instance.