DOCTORAL EXAM - TOPOLOGY - AUGUST 1999

You should attempt all the problems. Partial credit will be given for serious efforts.

1. Let M be the surface obtained by identifying the sides of an octagon according to the pattern $acbc^{-1}a^{-1}db^{-1}d^{-1}$. Compute the Euler characteristic of this surface and state whether or not the surface is orientable. Then use this information to determine what surface this is.

2. Let M be a closed, connected, orientable surface of genus two. Let C be a simple closed curve on M. Let D be a disk and $f : \partial D \to C$ a homeomorphism. Let X be the space obtained by attaching D to M using the map f. List all the possibilities for $\pi_1(X)$. Be sure to identify each group; do not just write down presentations.

3. Let S^1 be the circle and P^2 the real projective plane. Prove that every continuous function $f : S^1 \to S^1$ which factors through P^2 must be inessential. (Hint: If $g : P^2 \to S^1$ what must the induced map on fundamental groups be?)

4. Let $p : \widetilde{X} \to X$ be a covering map, where both spaces are path connected. Let $\tilde{x}_0 \in \widetilde{X}$ and $x_0 = p(\tilde{x}_0)$. Let $G = \pi_1(X, x_0)$ and $H = p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$. Exhibit a bijection from $p^{-1}(\tilde{x}_0)$ to the set G/H of all right cosets $H\alpha$ of H in G. Prove that it is well defined and is a bijection.

5. Let S^1 be the unit circle in the complex plane. Let D_1 and D_2 be two copies of the unit disk. Let f_1 , $f_2 : S^1 \to S^1$ be given by $f_1(z) = z$ and $f_2(z) = z^2$. Let X be the space obtained by attaching D_1 and D_2 to S^1 using f_1 and f_2, respectively. Compute $H_p(X)$ for all $p \geq 0$.

6. Suppose A is a retract of X. Prove that $H_p(X)$ is isomorphic to $H_p(A) \oplus H_p(X, A)$ for all $p \geq 0$.

7. Suppose X is a space such that $H_0(X) = \mathbf{Z}$, $H_1(X) = \mathbf{Z_5}$, $H_2(X) = \mathbf{Z_4}$, and $H_p(X) = 0$ for $p > 2$.

   (a) Calculate $H_p(X; \mathbf{Z_6})$ for all $p \geq 0$.

   (b) Calculate $H^p(X; \mathbf{Z})$ for all $p \geq 0$.

   ($\mathbf{Z_n}$ denotes the cyclic group of order n.)

8. Let M be a compact, connected, orientable 4-manifold (without boundary). Suppose $H_1(M) = \mathbf{Z_2}$ and $H^2(M) = \mathbf{Z} \oplus \mathbf{Z}$. Find $H_p(M)$ and $H^p(M)$ for all $p \geq 0$.