MASTER'S EXAM - TOPOLOGY - JUNE 1999

1. Let A and B be subsets of a topological space X. For each of the following inclusions give either a proof or a counterexample.
   (a) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
   (b) $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$

2. Let A, X, and Y be topological spaces. Give $X \times Y$ the product topology. Define the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ by $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$. Prove that a function $f : A \to X \times Y$ is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.

3. Give $\mathbf{R}^n$ the square metric $\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i|, 1 \leq i \leq n\}$. Let A be a closed subset of $\mathbf{R}^n$ which is bounded with respect to ρ. Prove that A is compact.

4. Let A and C be subsets of a topological space X. Give C the subspace topology. Prove that if C is connected, $C \cap A \neq \emptyset$, and $C \cap (X - A) \neq \emptyset$, then $C \cap \operatorname{Bd} A \neq \emptyset$. (If you wish, you may use any lemmas you like about $\operatorname{Int} A$ and $\operatorname{Bd} A$, but be sure to explicitly state them.)

5. Let $\mathbf{R}_\ell$ denote the real numbers in the lower limit topology. Prove that $\mathbf{R}_\ell$ is separable.