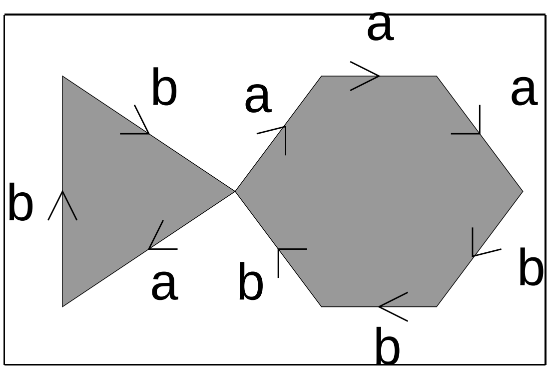
DOCTORAL EXAM - TOPOLOGY - JUNE 1999

1. Let X be the 2-complex given by performing the identifications indicated below. Write down a presentation for $\pi_1(X)$. You do not have to give a rigorous Seifert van-Kampen theorem argument to justify your presentation. The group is a familiar group; determine which familiar group it is.



2. Prove that every map $f : S^2 \to S^1$ is homotopic to a constant map. (Hint: universal covering space.)

3. Let S be the closed, connected, orientable surface of genus two. Let D_0, D_1, and D_2 be disjoint disks in S. Let X be the space obtained by removing the interiors of D_0, D_1, and D_2 and then gluing ∂D_1 to ∂D_2 by a homeomorphism. Find the Euler characteristic $\chi(X)$. What is $\pi_1(X)$?

4. Let $p : \widetilde{X} \to X$ be a covering map, where both spaces are path connected. Let $\tilde{x}_0 \in \widetilde{X}$ and $x_0 = p(\tilde{x}_0)$. Let A be a path connected subset of X such that $x_0 \in A$. Let $i : A \to X$ be the inclusion map. Prove that if $p^{-1}(A)$ is path connected, then for every $\beta \in \pi_1(X, x_0)$, $i_*(\pi_1(A, x_0))$ has non-empty intersection with the right coset $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))\beta$ of $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

5. Let $\phi^p \in C^p(X)$ and $\psi^q \in C^q(X)$. Let $\phi^p \smile \psi^q$ denote the cup product of these two cochains. State the formula expressing the coboundary $\delta(\phi^p \smile \psi^q)$ in terms of ϕ^p, ψ^q, $\delta\phi^p$, and $\delta\psi^q$. (You do not have to prove the formula.) Then use it to prove that $\{\phi^p\} \smile \{\psi^q\} = \{\phi^p \smile \psi^q\}$ gives a well defined function $H^p(X) \times H^q(X) \to H^{p+q}(X)$.

6. Suppose X is a space such that $H_0(X) = \mathbf{Z}$, $H_1(X) = \mathbf{Z_4}$, and $H_p(X) = 0$ for $p > 1$. Suppose Y is a space such that $H_0(Y) = \mathbf{Z}$, $H_1(Y) = \mathbf{Z_6}$, and $H_p(Y) = 0$ for $p > 1$. Calculate $H_p(X \times Y)$ for all $p \geq 0$. ($\mathbf{Z_n}$ denotes the cyclic group of order n.)

7. Let M be a connected n-manifold (without boundary). Let $x \in M$. Prove that $H_n(M, M - x)$
$\cong \mathbf{Z}$ and $H_p(M, M - x) = 0$ for $p \neq n$. Do not assume M has a triangulation. (Hints: Excision and homotopy axioms.)

8. Suppose M is a compact, orientable, connected 4-manifold (without boundary). $H_p(M) = A_p \oplus T_p$, where A_p is a free abelian group of finite rank and T_p is a finite abelian group. Prove that if $H_1(M) = 0$, then $A_1 \cong A_3$ and $T_2 = 0$. (Hints: Poincaré Duality and Universal Coefficient Theorem. Recall that every finite abelian group is a direct sum of cyclic groups.)