Masters Exam

Complex Analysis

January 1989

Instructions: Do any five of the following problems.

1. Determine whether

$$u(x,y) = x^3 - 3xy^2 + x^2 - y^2$$

is the real part of an entire function. If so, construct the entire function.

2. Determine whether or not an entire function $f(z)$ exists satisfying the following conditions. If there is, give an example. If not, explain why.

(a) $f(0) = 1, f(i) = 0$, and $|f(z)| \leq 6$ for all $z$.

(b) $f(z) = 0$ for all $z = n\pi i,\ n = 0, \pm 1, \pm 2, \pm 3, ....$

(c) $|f(i)| = 2$ and $|f(z)| = 1$ for all $z$ with $|z| = 2$.

3. For each of the following functions determine whether $z = 0$ is a removable singularity, an essential singularity, or a pole. If it is a pole, give its order. In all cases, compute the residue at $z = 0$.

(a) $\dfrac{(1+z)\ \sin(z)}{z^2}$ (b) $\cosh(\dfrac{3}{z^2})$ (c) $\dfrac{\cosh(z^2) - 1}{z^3}$

4. Find a series expansion (Taylor or Laurent) for the given function centered at $z_0$, valid in a neighborhood of $z_0$ (possibly with $z_0$ removed). Determine the domain of convergence of the series.

(a) $\dfrac{1}{(z-1)^3 z}$ centered at $z_0 = 1$. (b) $\sinh(z)$ centered at $z_0 = \frac{\pi i}{2}$.

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5. Compute one of the following two integrals:

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)(x^2+4)}\,dx$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)}\,dx$

6. (a) Suppose that $f(z)$ is analytic on and inside the circle of radius $R$ centered at $z_0$. Let

$$M(R) = \max_{|z-z_0|=R} |f(z)|$$

be the maximum of $f$ on this circle. Use the Cauchy integral formulas to show that

$$|f^{(n)}(z_0)| \leq \frac{n!\, M(R)}{R^n}.$$

(b) Show that if $f(z)$ is an entire function which satisfies $|f(z)| \leq |z|^M$ for a fixed positive integer $M$ then $f(z)$ must be a polynomial of degree at most $M$.

7. (a) Write down the linear fractional transformation $T$ mapping $\mathbf{D} = \{z \ : \ |z| < 1\}$ to $\mathbf{H} = \{w \ : \ Im(w) > 0\}$ and normalized so that $T(0) = i,\ T(i) = \infty$, and $T(-i) = 0$.

(b) What is the image of $\mathbf{H}$ under this mapping?

(c) Is this mapping conformal on $\mathbf{D}$? Explain why or why not.