Masters Exam : Complex Analysis

January 1991

Instructions: Do any 5 of the following problems.

1. Consider the many valued function $f(z) = (iz+1)^{1/2}$. Choose a branch of this function so that it becomes single valued and analytic and sketch its domain. For this branch, what is $f'(z)$?

2. Determine whether the following function is entire or not. If so, compute its derivative.

$$f(z) = e^{-y}((x+1)\cos(x) - y\sin(x)) + ie^{-y}((x+1)\sin(x) + y\cos(x))$$

3. Let C be the contour $C = \{z : |z| = 2\}$ with its positive orientation (i.e. counter clockwise). Compute the following integrals.

(a) $\displaystyle\int_C \frac{z+1}{\cos(z)}\,dz$ (b) $\displaystyle\int_C \frac{\cos(z)}{z^2}\,dz$ (c) $\displaystyle\int_C \frac{\sinh(z)}{z^6}\,dz$

4. State and prove Liouville's Theorem.

5. Let $f(z) = z^4(e^{1/z} - 1)$. Find all singularities of $f(z)$ and write down the principal part of $f(z)$ at each singular point. Determine whether each singularity is removable, essential, or a pole. Give the residue at each singularity.

6. A non-singular matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d, \in \mathbb{R}$, defines a linear fractional transformation by $T(z) = \dfrac{az+b}{cz+d}$. Let $\mathfrak{H} = \{z : \Im(z) > 0\}$ be the upper half plane. Show that the linear fractional transformation defined by the matrix T preserves $\mathfrak{H}$ iff $\det(T) > 0$. For which matrices T does the linear fractional transformation satisfy $T(z) = z$ for all $z \in \mathfrak{H}$?

7. Compute one of the following two integrals.

(a) $\displaystyle\int_0^\infty \frac{dx}{x^{1/2}(x^2+1)}$ (b) $\displaystyle\int_0^\infty \frac{x\sin(x)}{(x^2+4)(x^4+1)}\,dx$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX