PH. D. COMPREHENSIVE EXAM
ALGEBRA
JANUARY, 1995

Notation: $\mathbf{Z}_n$ denotes the integers modulo n.

1. (It is permissible to use the Sylow theorems.)
   (a) Show that every group of order p^n, where p is a prime number, is solvable.
   (b) Show that every group of order p^2q^2, where p and q are prime numbers with $p > q \geq 3$, is solvable.

2. Explicitly construct a subgroup H of $G = \mathbf{Z}_{27} \oplus \mathbf{Z}_3$ such that $G/H \simeq \mathbf{Z}_9$ or show that no such subgroup exists.

3. Let M be a Noetherian module over the ring R and let $\phi : M \to M$ be a surjective R-module homomorphism. Show that ϕ is an isomorphism.

4. Let R be a ring with identity $1 \neq 0$. Let J be the intersection of all the maximal left ideals of R. Show that $x \in J$ if and only if $1 - rx$ is a unit of R for all $r \in R$.

5. Factor the polynomial $x^4 - 2x^2 - 15$ into irreducible factors over the fields $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}_7$, $\mathbf{Z}_{19}$ and compute its Galois group over each of these fields.

6. Compute the Galois group of $x^4 + 2$ over $\mathbf{Q}$.