MASTER'S COMPREHENSIVE EXAM *NUMERICAL ANALYSIS*

JANUARY 1997

INSTRUCTIONS: Calculator required. Show eight or more figures in numerical results.

1. For cubic Lagrange interpolation at equidistant points $x_i = x_0 + ih$, $i = 0, 1, 2, 3$, one obtains an error estimate of the following form in the central interval $x_1 < x < x_2$:

$$(*) \qquad |f(x) - p_3(x)| \leq \mathbf{K}\, h^4 \max_{x_0 \leq \xi \leq x_3} |f^{(4)}(\xi)|, \quad \text{if } x_1 < x < x_2.$$

Here, $p_3(x)$ is the cubic Lagrange interpolation polynomial of the function $f(x)$, such that $p_3(x_i) = f(x_i)$ for $i = 0, 1, 2, 3$. Determine the smallest value of $\mathbf{K}$ for which $(*)$ is valid, and prove the validity of $(*)$ for the value of $\mathbf{K}$ that you have found.

2. Solve approximately the equation $f(x) \equiv x^2 - \cos x = 0$, in the interval $0.5 \leq x \leq 1.0$, by means of the Simplified Newton method

$$x_{n+1} = g(x_n), \quad g(x) = x - f(x)/m, \quad m = f'(x_0), \quad x_0 = 0.75.$$

a) For the given iteration function $g(x)$ determine a contraction constant λ, such that

$$1 > \lambda \geq \max_{0.5 \leq x \leq 1.0} |g'(x)|.$$

b) Iterate twice, obtaining x_1, and x_2.

c) Determine α and β, accurate bounds for the root $x_* = \lim_{n \to \infty} x_n$ of the equation $f(x) = 0$ such that $0.5 < \alpha < x_* < \beta < 1.0$. The bounds must be as accurate as possible, incorporate the numerical data from parts **a)** and **b)** above, and be based on an error estimate which is a corollary to the Contraction Mapping Theorem.

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3. The **Schröder series** iteration for solving $f(x) = 0$, $f \in \mathbf{C}^\infty(\mathbf{R})$, is defined by

$$(*) \qquad x_{n+1} = g(x_n), \quad g(x) = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2 f''(x)}{2[f'(x)]^3}.$$

a) State a Theorem to the effect that Schröder series iteration $(*)$ has at least cubic order of convergence. Be precise and include all the needed definitions and conditions.

b) Prove the Theorem that you stated in part **a)**.

4. Given is a matrix $A \in \mathbf{R}^{m\times n}$ and a factorization $A = U\,S\,V^T$, where the factors $U \in \mathbf{R}^{m\times k}$, $S \in \mathbf{R}^{k\times k}$, and $V \in \mathbf{R}^{n\times k}$ satisfy the conditions

(i) $U^T U = I_k$, the kxk identity matrix;

(ii) $V^T V = I_k$;

(iii) S is invertible.

a) Let $B = V\,S^{-1}\,U^T \in \mathbf{R}^{n\times m}$ and verify that

(i) $A\,B\,A = A$,

(ii) $B\,A\,B = B$,

(iii) $A\,B = (A\,B)^T$,

(iv) $B\,A = (B\,A)^T$.

b) Define column vector norms by $\|\mathbf{w}\|^2 = \mathbf{w}^T\mathbf{w}$, and for $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{y} \in \mathbf{R}^m$ show that

$$(I) \;\; \|\mathbf{x}\|^2 = \|(I_n - BA)\mathbf{x}\|^2 + \|BA\mathbf{x}\|^2, \qquad (II) \;\; \|\mathbf{y}\|^2 = \|(I_m - AB)\mathbf{y}\|^2 + \|ABy\|^2.$$

5. Apply the Composite Trapezoidal Rule on N subintervals,

$$T_N = h\left(\frac{1}{2}f(a) + \sum_{i=1}^{N-1} f(a+ih) + \frac{1}{2}f(b)\right), \quad h = \frac{b-a}{N},$$

to approximate $\mathcal{J} = \displaystyle\int_0^{2\pi} \ln(2+\cos x)\,dx$.

a) Explain why "the Euler-Maclaurin sum formula implies that $T_N \underset{(N\to\infty)}{\to} \mathcal{J}$ very rapidly."

b) Verify empirically the expected high accuracy of T_N by computing both, T_N and T_{2N}, for small N ($N = 4$ or $N = 8$; recall the recursive procedure in the Romberg Table).