Ph.D Comprehensive Exam: TOPOLOGY January 1997

Part I: The first part is true or false problems or statements. Support your answer by some short argument or explicit example. Typical answer should look like: *true, since every closed bounded subset of* $\mathbf{R}^n$ *is compact.*

1. Every covering space of a compact surface is compact.
2. The free product of two Abelian groups is Abelian.
3. The torus T^2 is a covering space of S^2.
4. There is a continuous function $r : S^2 \to S^1$ such that $r(x, y, 0) = (x, y)$ for all $(x, y) \in S^1$.
5. Any continuous function $f : D^2 \to D^2$ has the property: $x = f(x)$ for some $x \in D^2$.

Part II: Solve **five** of the following problems. Give detailed solutions for credits.

1. If the quotient space X/R is Hausdorff, then R is closed in the product space $X \times X$.
2. (a) Identify the surface whose triangulation is given by the following.

   $$\begin{array}{ccccc} 124 & 235 & 156 & 236 & 134 \\ 245 & 135 & 126 & 346 & 456 \end{array}$$

   (b) For any triangulation of a compact surface, show that

   $$2v \geq 7 + \sqrt{49 - 24\chi},$$

   where v is the total number of vertices and χ is the Euler number of the surface.
3. If Y_1 and Y_2 are connected closed 3-manifolds, then the fundamental group of connected sum $\pi_1(Y_1 \# Y_2)$ is the free product of $\pi_1(Y_1)$ and $\pi_1(Y_2)$.
4. (a) List all connected covering spaces of S^1.

   (b) If X is simply connected and $(\tilde{X}, p)$ is a covering space of X, then $p : \tilde{X} \to X$ is a homeomorphism.
5. Prove that there is no continuous map $f : S^2 \to S^1$ such that $f(-x) = -f(x)$.
6. Let X be a path-connected and Hausdorff space with a fixed point free involution $i : X \to X$ (i.e. an homeomorphism with $i \circ i = id_X$ and $i(x) \neq x$ for each point $x \in X$). Let $\pi : X \to Y$ be the projection from X onto the quotient space $Y = X/\sim$, where $x_1 \sim x_2$ if and only if $x_1 = x_2$ or $x_1 = i(x_2)$. Show that $\pi : X \to Y$ is a covering map.