DOCTORAL EXAM (OLD OPTION) - TOPOLOGY - JANUARY 1998

1. Prove that $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x,x)|x \in X\}$ is closed in $X \times X$.

2. Let $\{C_n\}$ be a sequence of closed, non-empty subsets of a compact space $X$ such that $C_n \supseteq C_{n+1}$ for all $n$. Prove from the definition of compactness that $\cap C_n \neq \emptyset$.

3. Let $U$ be a connected, non-empty open subset of $\mathbf{R}^2$. Prove that $U$ is path-connected.

4. Prove that every separable metric space satisfies the second axiom of countability.

5. Give examples of the following. Support your answers.

   (a) A space with trivial fundamental group.

   (b) A space with a finite non-trivial fundamental group.

   (c) A space with an infinite cyclic fundamental group.

   (d) A space with a non-abelian fundamental group.

6. For $i = 1, 2$, let $F_i$ be the free group on the set $S_i$ with respect to the function $\varphi_i : S_i \to F_i$. Assume that $S_1$ and $S_2$ are disjoint. Let $S = S_1 \cup S_2$. Let $F$ be the free product of $F_1$ and $F_2$ with respect to the homomorphisms $\psi_i : F_i \to F$. Define $\varphi : S \to F$ by $\varphi|_{S_i} = \psi_i \circ \varphi_i$. Prove that $F$ is the free group on $S$ with respect to $\varphi$.

7. Let $\mathcal{T}$ be a triangulation of $S^2$ such that each vertex has valence either 5 or 6. (Recall that the valence is the number of edges meeting the vertex.) Prove that $\mathcal{T}$ has exactly 12 vertices of valence 5.

8. Let $T$ be the torus $S^1 \times S^1$.

   (a) Give an example of a covering map $p : X \to T$ such that $\pi_1(X)$ is isomorphic to $\mathbf{Z}$.

   (b) Give an example of a 3-sheeted covering map $p : Y \to T$ with $Y$ connected.

   (c) Up to isomorphism how many 3-sheeted covering maps $p : Y \to T$ with $Y$ connected are there? Justify your answer.