DOCTORAL EXAM (NEW OPTION) - TOPOLOGY - JANUARY 1998

1. Give examples of the following. Support your answers.

   (a) A space with trivial fundamental group.

   (b) A space with a finite non-trivial fundamental group.

   (c) A space with an infinite cyclic fundamental group.

   (d) A space with a non-abelian fundamental group.

   For $i = 1, 2$, let F_i be the free group on the set S_i with respect to the function $\varphi_i : S_i \to F_i$. Assume that S_1 and S_2 are disjoint. Let $S = S_1 \cup S_2$. Let F be the free product of F_1 and F_2 with respect to the homomorphisms $\psi_i : F_i \to F$. Define $\varphi : S \to F$ by $\varphi|_{S_i} = \psi_i \circ \varphi_i$. Prove that F is the free group on S with respect to φ.

2. Let $\mathcal{T}$ be a triangulation of S^2 such that each vertex has valence either 5 or 6. (Recall that the valence is the number of edges meeting the vertex.) Prove that $\mathcal{T}$ has exactly 12 vertices of valence 5.

3. Let T be the torus $S^1 \times S^1$.

   (a) Give an example of a covering map $p : X \to T$ such that $\pi_1(X)$ is isomorphic to $\mathbf{Z}$.

   (b) Give an example of a 3-sheeted covering map $p : Y \to T$ with Y connected.

   (c) Up to isomorphism how many 3-sheeted covering maps $p : Y \to T$ with Y connected are there? Justify your answer.

4. (a) Compute the integral homology groups of $S^2 \times S^7$.

   (b) Suppose X is a space such that $H_0(X; \mathbf{Z}) = \mathbf{Z}$, $H_1(X; \mathbf{Z}) = \mathbf{Z_3}$, $H_2(X; \mathbf{Z}) = 0$, $H_3(X; \mathbf{Z}) = \mathbf{Z}$, and $H_n(X; \mathbf{Z}) = 0$ for $n > 3$. Calculate the homology and the cohomology groups of X with $\mathbf{Z_3}$ coefficients.

5. Let T be the torus $S^1 \times S^1$. Let z_0 be a point on S^1, and let c be the curve $S^1 \times \{z_0\}$. Let X be the space obtained by attaching a disk D to T so that ∂D is homeomorphically identified with c and $D \cap T = c$. Compute the integral homology groups of X.

6. Suppose M is a closed, orientable, connected 3-manifold. (Recall that "closed" in this context means "compact and without boundary.") Prove that $\chi(M) = 0$. Hint: Duality.