MASTERS COMPREHENSIVE EXAM—COMPLEX VARIABLES—January 1999

1. Use the definition of the derivative to prove that $f'(z) = -\dfrac{1}{z^2}$ if $f(z) = \dfrac{1}{z}$ $(z \neq 0)$.

2. Define $f(z) = \dfrac{\sin z}{z}$ if $z \neq 0$, and define $f(0) = 1$. Prove that f is an entire function.

3. Let C be the unit circle taken counterclockwise, and define

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\, dz$$

for $|w| \neq 1$.

(a) Compute $g(\frac{1}{2})$.

(b) Compute $g(2)$.

4. Let $f(z) = e^{1/z}$, and let C be the unit circle taken counterclockwise.

(a) Determine the Laurent series of $f(z)$ at $z = 0$.

(b) Determine the residue of $f(z)$ at $z = 0$.

(c) Is $z = 0$ a removable singular point, a pole, or an essential singular point of $f(z)$?

(d) Evaluate the contour integral of $z^2 f(z)$ along C.

5. Let $a > 1$. Using residue theory, prove that

$$\int_0^{\pi} \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}}.$$