MASTER'S COMPREHENSIVE EXAMS *NUMERICAL ANALYSIS*

JANUARY 1999

INSTRUCTIONS: Calculator required. Show eight or more decimal figures. Two problems completely solved will be sufficient for a passing grade.

1. Let ϵ and K be positive numbers with $0 < K < 1$ and $L = \frac{1}{1-K}$. Prove the following **Lemma:** Given (real or complex) numbers δ_j, $j = 1, \ldots, n$, with $|\delta_j| \leq \epsilon$, $j = 1, \ldots, n$, define α_n by the equation $1 + \alpha_n = \prod_{j=1}^{n}(1 + \delta_j)$. If $n\epsilon \leq K$ then $|\alpha_n| \leq L\, n\epsilon$. Hint: Use mathematical induction. The Lemma, which is used for bounding roundoff errors, is true in the stronger version with $L = 1 + K$ but the proof is slightly more difficult.

2. Solve approximately the equation $x = g(x)$ for the function

$$g(x) = \ln x - \frac{x}{3} + 3$$

in the interval $2 \leq x \leq 4$ by means of the iteration $x_{n+1} = g(x_n),\ n = 0, 1, \ldots,$ with $x_0 = 3.0$

a) Compute explicitly $x_1,\ x_2,\ x_3$.

b) Apply the contraction mapping theorem (or, its proof and corollaries) to obtain an accurate upper bound for the error $|x_3 - x_*|$, where x_* is the exact solution of the equation $x = g(x)$ in the interval.

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3. Let $f(x)$ be a sufficiently differentiable function on $[a, b]$.

a) State the composite trapezoidal rule for $\int_a^b f(x)\,dx$.

b) Give a formula for the error. Explain.

c) Use the composite trapezoidal rule to estimate $\int_0^1 e^{-x^2}\,dx$ to within 0.005.

d) For the value of h found in part **c)** above, calculate an asymptotic error estimate. Use this to give a corrected value (this is sometimes referred to as the *corrected trapezoidal rule*).