HOMEWORK 1
ALGEBRA-II, MATH 5623, SPRING 2008

All rings have 1, and all modules are unitary.

Exercise 1. Show that the following conditions on a ring R are equivalent:

(1) Every R-module is injective.
(2) Every R-module is projective.
(3) Every short exact sequence of R-modules is split.

Give an example of a ring with these properties, and also an example of a ring which does not satisfy any of these properties.

Exercise 2. Let R be a ring, and let M be an R-module. Show that there is an exact sequence

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

where each P_i is a projective module. Such a sequence is called a *projective resolution* of M. Further, show that one can arrange for each P_i to be a free module, in which case the exact sequence is called a *free resolution* of M.

Exercise 3. Let R be a ring, and let M be an R-module. Show that there is an exact sequence

$$0 \longrightarrow M \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_n \longrightarrow \cdots$$

where each I_i is an injective module. Such a sequence is called an *injective resolution* of M.

Exercise 4. Show that

(1) A finite abelian group is not divisible.
(2) A free abelian group is not divisible.
(3) $\mathbb{Q}$ is a divisible group.
(4) For a prime p, the subgroup $\mathbb{Z}[1/p]/\mathbb{Z}$ of $\mathbb{Q}/\mathbb{Z}$ is a divisible group. (Recall that $\mathbb{Z}[1/p]$ is all rational numbers whose denominators are p-powers.)

Exercise 5. Show that

(1) A quotient of a divisible group is divisible.
(2) A direct summand of a divisible group is divisible.
(3) A direct sum of divisible groups is divisible.

Give an example of a non-divisible subgroup of a divisible group.

Exercise 6. If R is a ring such that every R-module is free, then show that R is a division ring.

Exercise 7 (Comprehensive, January 2008)**.** Let B be a subgroup of an abelian group A. Let $f : B \to \mathbb{Q}$ be a homomorphism of abelian groups. Show that there is a homomorphism $F : A \to \mathbb{Q}$ such that $F(b) = f(b)$ for all $b \in B$. (Here $\mathbb{Q} = \mathbb{Q}^+$ is the additive group of all rational numbers.)