HOMEWORK 2
ALGEBRA-II, MATH 5623, SPRING 2008

Exercise 1. In the category of R-modules, given homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, show that the following are equivalent:

(1) $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is split exact.
(2) $0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$ is split exact for all modules M.
(3) $0 \to \mathrm{Hom}_R(C, N) \to \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N) \to 0$ is split exact for all modules N.

Exercise 2. In the category of R-modules, prove or disprove

$$\mathrm{Hom}_R(M, \bigoplus_i N_i) \simeq \bigoplus_i \mathrm{Hom}_R(M, N_i).$$

Exercise 3. Let R be a ring. Define the *opposite ring* R^{op} as the ring whose underlying set and additive group is the same as R and for $x, y \in R$ their product $x * y$ in R^{op} is defined by $x * y = yx$. Show that

(1) A left (resp. right) R-module is the same as a right (resp. left) R^{op}-module.
(2) $R^{\mathrm{op}} \simeq \mathrm{Hom}_R(R, R)$, where Hom_R denotes all homomorphisms as left R-modules.
(3) Let F be a field, and let $R = M_n(F)$. Show that R^{op} is isomorphic with R.

Exercise 4. If P is a finitely generated projective module, then show that so is P^*. Further, show that the canonical map $P \to P^{**}$ given by evaluation is an isomorphism. (A module M with the property that $M \simeq M^{**}$ is said to be *reflexive*.)

Exercise 5. Let V be a vector space. Let S be a subset of V. The annihilator S^0 of S is defined by

$$S^0 = \{f \in V^* : f(s) = 0, \forall s \in S\}.$$

(1) Show that S^0 is a subspace of V^*.

Now assume that V has finite dimension, and let W be a subspace of V.

(2) Show that $\dim(W^0) = \dim(V) - \dim(W)$.
(3) Show that $W^* \simeq V^*/W^0$.
(4) Under the canonical isomorphism $V \to V^{**}$, show that $W \simeq W^{00}$.

In the literature, W^0 is also denoted as $W^\perp$.

Exercise 6. Let F be a field and let V be an infinite-dimensional F-vector space. Show that

$$\dim_F(V) < \dim_F(V^*).$$

Exercise 7. In the category of $\mathbb{Z}$-modules, show that

(1) $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, A) \simeq A[n]$. (Here $A[n] := \{a \in A : na = 0\}$ denotes the n-torsion of A.)
(2) $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/(n, m)\mathbb{Z}$.
(3) $(\mathbb{Z}/n\mathbb{Z})^* = (0)$.

Exercise 8 (Bonus exercise)**.** Show that an infinite product of copies of $\mathbb{Z}$ is not a free $\mathbb{Z}$-module.