MATH 2153-006 MIDTERM-I
FEBRUARY 17, 2009

SOLUTIONS.

All questions are worth ten points. The maximum possible total is 70. You have approximately an hour and 15 minutes for this exam. Calculators, cell phones, i-pods and other technological gizmos are not allowed!

Question	Marks
Total	

Question 1. Evaluate the integral

$$\int \sin^{-1} x \, dx.$$

Integrate by parts. Put $u = \sin^{-1}(x)$ $\quad dv = dx$

$du = \dfrac{1}{\sqrt{1-x^2}} dx$ $\quad v = x.$

$$\int \sin^{-1} x \, dx = \int u \, dv = uv - \int v \, du = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx.$$

For $\int \frac{x}{\sqrt{1-x^2}} dx$, put $t = 1 - x^2$, $dt = -2x dx$.

$$\therefore \int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = = -\frac{1}{2} \cdot 2\sqrt{t} = -\sqrt{t} = -\sqrt{1-x^2}$$

$$\therefore \boxed{\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C.}$$

Question 2. Evaluate the integral

$$\int_0^{\pi/2} \cos^3 x \, dx.$$

Put $u = \sin x$, $\quad x = 0 \Rightarrow u = 0$

$du = \cos x \, dx \quad x = \pi/2 \Rightarrow u = 1$

$$\int_0^{\pi/2} \cos^3 x \, dx = \int_0^1 \cos^2 x \cdot du = \int_0^1 (1-u^2)\, du = \left[u - \frac{u^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

Answer $= \frac{2}{3}$

Question 3. Evaluate the integral

$$\int \frac{x^3}{\sqrt{x^2+25}} \, dx.$$

Put $x = 5\tan\theta$ $\quad \sqrt{x^2+25} = \sqrt{25\tan^2\theta + 25} = \sqrt{25\sec^2\theta} = 5\sec\theta.$

$dx = 5\sec^2\theta \, d\theta$

$$\int \frac{x^3}{\sqrt{x^2+25}}\, dx = \int \frac{5^3 \tan^3\theta}{5\sec\theta} \cdot 5\sec^2\theta \, d\theta = 5^3 \int \tan^3\theta \sec\theta \, d\theta.$$

Put $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$, $\tan^2\theta = \sec^2\theta - 1 = u^2 - 1$

$$= 5^3 \int \tan^2\theta \cdot du = 5^3 \int (u^2-1)\, du = 5^3\left(\frac{u^3}{3} - u\right) + C$$

$$= 5^3\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C = \frac{5^3}{3}\sec^3\theta - 5^3\sec\theta + C$$

$\tan\theta = \frac{x}{5}$

$\sec\theta = \sqrt{1 + \frac{x^2}{25}} = \frac{\sqrt{x^2+25}}{5}$

$= (x^2+25)^{1/2}$

$$= \frac{5^3}{3}\frac{(x^2+25)^{3/2}}{5^3} - 5^3 \frac{(x^2+25)^{1/2}}{5} + C$$

$$= \boxed{\frac{(x^2+25)^{3/2}}{3} - 25(x^2+25)^{1/2} + C}$$

Answer :- $\boxed{\dfrac{1}{x^4-x} = -\dfrac{1}{x} + \dfrac{1}{3(x-1)} + \dfrac{2x+1}{3(x^2+x+1)}}$

Question 4. Write down the partial fraction expansion of

$$\frac{1}{x^4-x}.$$

$$\frac{1}{x^4-x} = \frac{1}{x(x^3-1)} = \frac{1}{x(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1}.$$

$$\Rightarrow 1 = A(x-1)(x^2+x+1) + B\cdot x\cdot(x^2+x+1) + (Cx+D)x\cdot(x-1).$$

Put $x=0$, $1 = -A \Rightarrow A = -1$

$x = 1$, $1 = 3B \Rightarrow B = 1/3$.

$x = 2$, $1 = 7A + 14B + 4C + 2D \Rightarrow 4C + 2D = 1 + 7 - \frac{14}{3} = \frac{10}{3}$

$x = -1$, $1 = -2A - B + (-C+D)2 \Rightarrow -2C + 2D = 1 - 2 + \frac{1}{3} = -\frac{2}{3}$.

$$\left.\begin{array}{l} 2C + D = 5/3 \\ -C + D = -1/3 \end{array}\right\} \Rightarrow \begin{array}{l} 3C = 6/3 = 2 \Rightarrow C = 2/3 \\ 3D = 1 \Rightarrow D = 1/3. \end{array}$$

Question 5. Evaluate the integral

$$\int \sqrt{\frac{1+x}{1-x}}\, dx.$$

(Hint: Try some algebraic manipulation of the integrand.)

$$\int \sqrt{\frac{1+x}{1-x}}\, dx = \int \frac{1+x}{\sqrt{1-x^2}}\, dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\, dx}{\sqrt{1-x^2}}$$

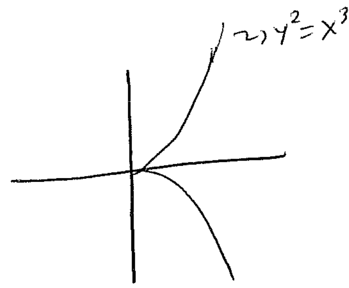
$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

$\int \frac{x\, dx}{\sqrt{1-x^2}}$ put $u = 1-x^2$, $du = -2x\, dx \Rightarrow x\,dx = -\frac{1}{2} du.$

$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\cdot 2\sqrt{u} = -\sqrt{u}. = -\sqrt{1-x^2}$$

$$\boxed{\int \sqrt{\frac{1+x}{1-x}}\, dx = \sin^{-1} x - \sqrt{1-x^2} + C.}$$

Answer:- Total arc length $= \frac{2}{27}\left(40^{3/2} - 13^{3/2}\right).$



Question 6. Find the length of the curve $y^2 = x^3$ for $1 \le x \le 4$.

$y^2 = x^3 \Rightarrow y = x^{3/2}, \quad 1 \le x \le 4, \quad \frac{dy}{dx} = \frac{3}{2}x^{1/2}$

Arc length $= \int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_1^4 \sqrt{1+\frac{9x}{4}}\,dx = \frac{1}{2}\int_1^4 \sqrt{4+9x}\,dx = \left[\frac{1}{2}\,\frac{(4+9x)^{3/2}}{\frac{3}{2}\cdot 9}\right]_1^4$

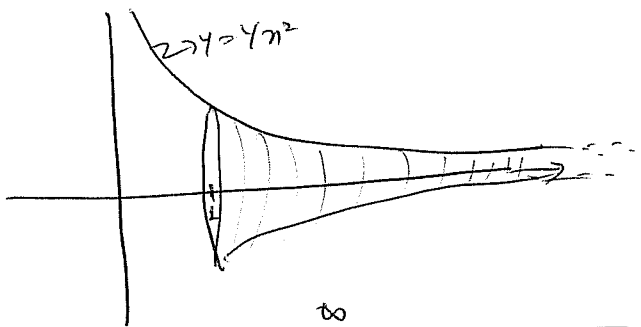
$= \left[\frac{(4+9x)^{3/2}}{27}\right]_1^4 = \frac{40^{3/2} - 13^{3/2}}{27}$

But this is only half the answer, because, $y^2 = x^3 \Rightarrow y = \pm x^{3/2} \Rightarrow$ There is an identical piece of the curve "below" the x-axis.

Question 7. Consider the surface obtained by rotating the curve

$$y = 1/x^2 \quad (x \ge 1)$$

about the x-axis. Is the area of this surface finite or infinite? Justify your answer.



$\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$

Surface Area $= \int_1^\infty 2\pi\cdot\frac{1}{x^2}\sqrt{1+\left(\frac{-2}{x^3}\right)^2}\,dx = 2\pi\int_1^\infty \frac{1}{x^2}\sqrt{1+\frac{4}{x^6}}\,dx.$

Now, if $x \ge 1$, $\frac{1}{x} \le 1 \Rightarrow \frac{1}{x^6} \le 1 \Rightarrow 1+\frac{4}{x^6} \le 5.$

$\therefore$ Surface Area $= 2\pi\int_1^\infty \frac{1}{x^2}\sqrt{1+\frac{4}{x^2}}\,dx \le 2\pi\int_1^\infty \frac{1}{x^2}\sqrt{5}\,dx$ (comparison Test)

$= 2\pi\sqrt{5}\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} 2\pi\sqrt{5}\int_1^t \frac{1}{x^2}\,dx$

$= \lim_{t\to\infty} 2\pi\sqrt{5}\left[\frac{-1}{x}\right]_1^t = \lim_{t\to\infty} 2\pi\sqrt{5}\left[1-\frac{1}{t}\right] = 2\pi\sqrt{5} < \infty$

$\Rightarrow$ Surface Area is finite