TOPICS IN GEOMETRY: SHEAF THEORY
MATH 6490, SPRING 2009
HOMEWORK 5

For all the exercises below we work in the category of (pre)sheaves of abelian groups over a topological space X.

Exercise 1. Show that a sequence

$$\cdots \to \mathcal{F}^{i-1} \to \mathcal{F}^{i} \to \mathcal{F}^{i+1} \to \cdots$$

of sheaves and morphisms is exact if and only if the corresponding sequence of stalks is exact (as a sequence of abelian groups) for every point of X.

Exercise 2. Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

Exercise 3. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

(1) Show that $\mathrm{im}(\phi) \simeq \mathcal{F}/\ker(\phi)$.
(2) Show that $\mathrm{coker}(\phi) \simeq \mathcal{G}/\mathrm{im}(\phi)$.

Exercise 4. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. (This is called the direct sum of $\mathcal{F}$ and $\mathcal{G}$, and is denoted $\mathcal{F} \oplus \mathcal{G}$.) Show that $\mathcal{F} \oplus \mathcal{G}$ plays the role of direct sum and of direct product of $\mathcal{F}$ and $\mathcal{G}$ in the category of sheaves of abelian groups on X.

Exercise 5. *(Extending a Sheaf by Zero.)* Let X be a topological space, Z a closed subset of X, and $U = X - Z$ the complementary open subset. Let $i : Z \to X$ and $j : U \to X$ be the inclusion maps.

(1) Let $\mathcal{G}$ be a sheaf on Z, and let $i_*\mathcal{G}$ be its direct image sheaf on X. For $x \in X$, show that the stalk $(i_*\mathcal{G})_x$ is $\mathcal{G}_x$ if $x \in Z$ and $(i_*\mathcal{G})_x = 0$ if $x \notin Z$. (This justifies the terminology that $i_*\mathcal{G}$ is obtained by extending $\mathcal{G}$ by 0 outside of Z.)
(2) Let $\mathcal{H}$ be a sheaf on U. Let $j_!\mathcal{H}$ be the sheaf on X associated to the presheaf: $V \mapsto \mathcal{H}(V)$ if $V \subset U$, and $V \mapsto 0$ if $V \not\subset U$. For $x \in X$, show that the stalk $(j_!\mathcal{H})_x$ is $\mathcal{H}_x$ if $x \in U$ and $(j_!\mathcal{H})_x = 0$ if $x \notin U$. (This justifies the terminology that $j_!\mathcal{H}$ is obtained by extending $\mathcal{H}$ by 0 outside of Z.) Explain why $j_!\mathcal{H}$ is not the same as the direct image sheaf $j_*\mathcal{H}$.
(3) Let $\mathcal{F}$ be a sheaf on X. Show that one has an exact sequence of sheaves on X:

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0.$$

* * * * * * *