TOPICS IN GEOMETRY: SHEAF THEORY
MATH 6490, SPRING 2009
HOMEWORK 7

Exercise 1. Let $0 \to \mathcal{F} \to G \to \mathcal{H} \to 0$ be a short exact sequence of sheaves. Assume that $\mathcal{F}$ is a flabby sheaf. For any open set U of X show that the sequence

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

of sections over U is exact.

Exercise 2. Let $0 \to \mathcal{F} \to G \to \mathcal{H} \to 0$ be a short exact sequence of sheaves. If $\mathcal{F}$ and $\mathcal{G}$ are flabby sheaves then show that $\mathcal{H}$ is also a flabby sheaf.

Exercise 3. Let $\mathcal{F}$ be a sheaf on a topological space X. Let $s \in \mathcal{F}(X)$. For $x \in X$, let $s(x)$ denote the germ at x determined by s. Define the support of s as

$$\mathrm{Supp}(s) = \{x \in X \ : \ s(x) \neq 0\}.$$

Show that $\mathrm{Supp}(s)$ is a closed subset of X. Give an example to show that $\mathrm{Supp}(s)$ is not in general open. Define the support of the sheaf $\mathcal{F}$ as

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \ : \ \mathcal{F}_x \neq 0\}.$$

Show by an example that $\mathrm{Supp}(\mathcal{F})$ need not be closed.

Exercise 4. Let $\mathcal{R}$ be a sheaf of commutative rings (with identities) on a topological space X. Assume that $\mathcal{R}$ has following property: For any open cover $\{U_i\}$ of X there exists a partition of unity subordinate to that cover, meaning that there exists global sections $h_i \in \mathcal{R}(X)$ such that: (i) $\mathrm{Supp}(h_i) \subset U_i$; (ii) for any $x \in X$ there are only finitely many indices i for which $h_i(x) \neq 0$; and (iii) $1 = \sum_i h_i$ in $\mathcal{R}(X)$.

Now let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules over X. This means that for each open set U, $\mathcal{F}(U)$ is an $\mathcal{R}(U)$-module, and for $V \subset U$ the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is an $\mathcal{R}(V)$-module homomorphism. Show that $\mathcal{F}$ is an acyclic sheaf. *(Hint: See Exercise 16 in Harder's book.)*

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