Lectures 11 & 12.

Basic Notions of Sheaves.

§1 Presheaf.

Definition Let X be a topological space.

A presheaf of abelian groups on X, denoted $\mathcal{F}$, consists of

- $\forall\ U$ open in X, an abelian group $\mathcal{F}(U)$
- $\forall\ U, V$ open, $V \subset U$, a homomorphism $r_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

such that

(i) $\mathcal{F}(\emptyset) = 0$.

(ii) $r_{UU} = 1_{\mathcal{F}(U)}$

(iii) $\forall\ U, V, W$, with $W \subset V \subset U$,

$$r_{UW} = r_{VW} \circ r_{UV}$$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "r_{UV}"] \arrow[dr, "r_{UW}"'] & \mathcal{F}(V) \arrow[d, "r_{VW}"] \\ & \mathcal{F}(W) \end{tikzcd}$$

- **Alternative definition**:-

(Recall that any top. space X determines a category $\mathcal{C}_X$:- $ob(\mathcal{C}_X) =$ open subsets, $\mathrm{Hom}_{\mathcal{C}_X}(U,V) = \begin{cases} \{\phi_{UV}\}, & U \subset V \\ \text{empty}, & U \not\subset V \end{cases}$)

A presheaf $\mathcal{F}$ of abelian groups is a contravariant functor $\mathcal{F} : \mathcal{C}_X \longrightarrow AB$.

- Comments:-

(i) We can talk about a presheaf of rings, modules, vector spaces ...etc.

(Unless otherwise mentioned, let us stick to (pre)sheaves of abelian groups

(ii) Elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.

Sometimes, $\mathcal{F}(U)$ is also denoted $\Gamma(U, \mathcal{F})$.

Elements of $\mathcal{F}(X)$ are called global sections.

<u>Definition</u> (Morphism of presheaves)

Let $\mathcal{F}, \mathcal{G}$ be presheaves on a top. space X.

A morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ of homomorphism (of abelian groups) such that $\forall\, U, V$ - open, $V \subset U$, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "r^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "r^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V) \end{tikzcd}$$

commutes.

<u>Definition</u>: PS_X = category of presheaves of abelian groups on X.

<u>Note</u>: If we take $\mathcal{F}, \mathcal{G}$ to be contravariant functors $C_X \longrightarrow AB$ then a morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ of presheaves is nothing but a natural transformation from $\mathcal{F}$ to $\mathcal{G}$.

§2 Sheaves.

Definition A *sheaf* $\mathcal{F}$ *on a top. space* X is a presheaf which satisfies:-

(i) If U is open, if $\{V_i\}$ is an open cover of U, if $s \in \mathcal{F}(U)$ is such that $s|_{V_i} = 0 \ \forall i$, then $s = 0$.

(A locally trivial section is trivial.)

(ii) If U is open, if $\{V_i\}$ is an open cover of U, given $s_i \in \mathcal{F}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, $\exists\, s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i$.

(Given local sections which patch up one can find a "global" section.)

Note:- The above two "sheaf axioms" can be rephrased as :- U - open, $U = \bigcup_i V_i$ open cover, the sequence of abelian groups

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(V_i) \longrightarrow \prod_{i,j} \mathcal{F}(V_i \cap V_j)$$
$$s \longmapsto (s|_{V_i})$$
$$(s_i) \longmapsto (s_i|_{V_i \cap V_j} - s_j|_{V_i \cap V_j}).$$

is an exact sequence.

Definition:- (Morphism of sheaves.) Let $\mathcal{F}, \mathcal{G}$ be sheaves on X. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves is a morphism of the underlying presheaves.

Definition:- S_X = category of sheaves (of abelian groups) on X.

Note:- S_X is, by definition, a full subcategory of PS_X.

§3. Examples.

Let X be a topological space.

Let A be an abelian group.

① The constant pre-sheaf on X determined by A :-

Let me denote this as ${}_{ps}\underline{A}$. (or maybe, ${}^{p}A_X$)

. $U \underset{\text{open}}{\subset} X \longmapsto {}_{ps}\underline{A}(U) = A$ if $U \neq \phi$ and ${}_{ps}\underline{A}(\phi) = 0$.

. $V \subset U \subset X$;

$$\begin{array}{ccc} {}_{ps}\underline{A}(U) & \xrightarrow{r_{UV}} & {}_{ps}\underline{A}(V) \\ \| & & \| \\ A & \xrightarrow{1_A} & A \end{array}$$

All restriction maps are 1_A. unless $V = \phi$.

Note:- This is a presheaf and is not a sheaf.

For example, if V_1, V_2 are disjoint open sets, $U = V_1 \amalg V_2$,

Let $a_i \in {}_{ps}\underline{A}(V_i)$, $a_1 \neq a_2$ then $V_1 \cap V_2 = \phi \implies a_1|_{V_1 \cap V_2} = a_2|_{V_1 \cap V_2}$

But $\nexists\, a \in {}_{ps}\underline{A}(U)$ s.t $a|_{V_i} = a_i$.

② The constant sheaf on X determined by A.

Let me denote this sheaf by A_X.

$U \underset{\text{open}}{\subset} X$, $A_X(U) =$ locally constant A-valued functions on U.

$V \subset U$, $A_X(U) \longrightarrow A_X(V)$ is simply restriction of functions.

(Equivalently, put the discrete topology on A,
$A_X(U) =$ continuous fns. from U to A.).

Check that this is indeed a sheaf.

③ (a) C^0_X = sheaf of continuous $\mathbb{R}$-valued functions on X.

$U \underset{open}{\subset} X$, $\quad C^0_X(U) = \{ f : U \longrightarrow \mathbb{R} \mid f \text{ is continuous} \}$.

$V \subset U \quad C^0_X(U) \longrightarrow C^0_X(V)$ is the restriction map of functions.

note: C^0_X is a sheaf of $\mathbb{R}$-vector spaces on X.

(b) $X = M$ - smooth manifold., C^∞_M = sheaf of smooth (infinitely differentiable) functions on M.

§4 Stalks of a (pre) sheaf :-

Definition:- Let $\mathcal{F}$ be a presheaf on a top. space X.

Let $x \in X$.

The stalk of $\mathcal{F}$ at x, denoted $\mathcal{F}_x$, is defined as

$$\mathcal{F}_x = \varinjlim_{x \in U \underset{open}{\subset} X} \mathcal{F}(U).$$

Note:- The set of open subsets of X containing x is ordered by reverse inclusion
$U \leq V$ if $U \supset V$. This forms a directed ~~system~~ set.

If $U \leq V$ then we have the restriction maps $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$.

Hence $\{\mathcal{F}(U)\}_{x \in U}$ is a direct system.

Any element of $\mathcal{F}_x$ is a pair (U, s) with $x \in U \underset{open}{\subset} X$ and $s \in \mathcal{F}(U)$

Two pairs (U, s) & (V, t) are equivalent under:-

$$(U, s) \sim (V, t) \iff s|_{U \cap V} = t|_{U \cap V}$$

Proposition :-

Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves on a topological space X. Then, φ is an isomorphism $\iff$ $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is an isomorphism $\forall x \in X$.

Pf :- If φ is an iso. $\Rightarrow$ φ_x is an iso. $\forall x$ is obvious!

- Assume now that φ_x is an isomorphism $\forall x \in X$.

WTS :- $U \underset{\text{open}}{\subset} X$, $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is an isomorphism.

- $\varphi(U)$ is injective :-

Let $s \in \mathcal{F}(U)$ and $\varphi(U)s = 0$ in $\mathcal{G}(U)$.

Let $x \in U$, then $\varphi_x(U, s) = (U, \varphi(U)s) = 0$ in $\mathcal{G}_x$ $\Rightarrow$ $(U, s) = 0$ in $\mathcal{F}_x$.

$\Rightarrow$ $\exists$ nbhd. V_x of x, s.t $(U, s) \sim (V_x, 0)$ or $s|_{V_x} = 0$.

$U = \bigcup_{x \in U} V_x$, $s|_{V_x} = 0$ $\forall x \in U$ $\Rightarrow$ $s = 0$ by the first sheaf axiom

$\Rightarrow$ $\varphi(U)$ is injective.

- $\varphi(U)$ is surjective.

Let $t \in \mathcal{G}(U)$. Let $x \in U$ and consider (U, t) as an element of $\mathcal{G}_x$.

Since $\varphi_x : \mathcal{F}_x \twoheadrightarrow \mathcal{G}_x$, $\exists$ an element of $\mathcal{F}_x$ mapping to (U, t).

i.e., $\exists$ nbhd V_x of x, $\exists\, s_x \in \mathcal{F}(V_x)$ s.t $\varphi_x(V_x, s_x) = (U, t)$. in $\mathcal{G}_x$

Clearly, $U = \bigcup_{x \in U} V_x$.

Observe that $s_x|_{V_x \cap V_y} = s_y|_{V_x \cap V_y}$.

because for any point z in $V_x \cap V_y$, both these sections give elements of $\mathcal{F}_z$ and are mapped to the same element of $\mathcal{G}_z$.

Hence $\exists\, s \in \mathcal{F}(U)$ s.t $s|_{V_x} = s_x$.

Check that $\varphi(U)s = t$.

(One is using the following fact about sheaves :- $s_1, s_2 \in \mathcal{F}(U)$ and $s_{1,x} = s_{2,x}$ $\forall x \in U$ then $s_1 = s_2$.)

§5 Sheafification of a presheaf.

Definition/Proposition:-
Let $\mathcal{F}$ be a presheaf on X.
There exists a sheaf $\mathcal{F}^{\#}$, a morphism $j : \mathcal{F} \longrightarrow \mathcal{F}^{\#}$ of presheaves such that for any sheaf $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$, there exists a unique morphism $\varphi^{\#} : \mathcal{F}^{\#} \longrightarrow \mathcal{G}$ such that $\varphi = \varphi^{\#} \circ j$.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\varphi"] \arrow[dr, "j"'] & & \mathcal{G} \\ & \mathcal{F}^{\#} \arrow[ur, "\varphi^{\#}"'] & \end{tikzcd}$$

Note:- We can also say that $\mathcal{F}^{\#}$ represents the functor $S_X \longrightarrow SETS$.
$\mathcal{G} \longmapsto \mathrm{Hom}_{PS_X}(\mathcal{F}, \mathcal{G})$, i.e., $\mathrm{Hom}_{PS_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{S_X}(\mathcal{F}^{\#}, \mathcal{G})$

By Yoneda's lemma, if $\mathcal{F}^{\#}$ exists, it is unique. up to unique isomorphism

$\mathcal{F}^{\#}$ is called the sheafification of $\mathcal{F}$.

Construction of $\mathcal{F}^{\#}$:-
For any open subset U of X,

$$\mathcal{F}^{\#}(U) = \left\{ s : U \longrightarrow \bigcup_{x \in U} \mathcal{F}_x \ : \ \begin{array}{l} \text{(i)} \ s(x) \in \mathcal{F}_x \\ \text{(ii)} \ \forall x \in U, \ \exists \text{ open set } V, \ x \in V \subset U, \ \exists t \in \mathcal{F}(V) \text{ such that} \\ \quad \text{for all } y \in V, \text{ the germ } t_y = s(y) \end{array} \right\}$$

Note:- $\mathcal{F}^{\#}$ is a sheaf because sections over U are determined locally.

- The morphism $j : \mathcal{F} \longrightarrow \mathcal{F}^{\#}$ is the "obvious map" $\mathcal{F}(U) \longrightarrow \mathcal{F}^{\#}(U)$.
If $s \in \mathcal{F}(U)$ then $s : U \longrightarrow \bigcup_{x \in U} \mathcal{F}_x$ by defining $s(x) =$ germ at x determined by the pair (U, s).

- Verifying the universal property is easy (but tedious to write!).

Example:-

Let A be an abelian group.

The sheafification of the constant presheaf ${}^{p}A_X$ (determined by A) is in fact the constant sheaf A_X. , i.e., ${}^{p}A_X^{\#} \cong A_X$

Example

Let $\mathcal{F}$ be a presheaf on X. & $\mathcal{F}^{\#}$ its sheafification.

Then $\forall\, x \in X$, $\mathcal{F}$ and $\mathcal{F}^{\#}$ have the same stalks:-

$$\mathcal{F}_x \simeq \mathcal{F}^{\#}_x.$$

"pf":- (It is best if you find the proof of such statements yourselves!)

- Any element of $\mathcal{F}_x = [(U,s)]$, $(U,s) \sim (U',s')$ if $s|_{U\cap U'} = s'|_{U\cap U'}$
- For germs of sections, we can always go down to a smaller open subset of X containing x.
- The map $j_x : \mathcal{F}_x \longrightarrow \mathcal{F}^{\#}_x$ may be written as:-

  $j_x[(U,s)] = [(U,\tilde{s})]$ when $\tilde{s} : U \longrightarrow \bigcup_{x\in U} \mathcal{F}_x$ is the function determined by s.

  i.e., $\tilde{s}(x) = s_x$ $\quad s_x = [(U,s)]$

- $[(U,\tilde{s})] = 0$ in $\mathcal{F}^{\#}_x$ $\Rightarrow$ $\exists\, V,\ x \in V \subset U,\ \tilde{s}(x) = 0_x$, $0 \in \mathcal{F}(V)$.

  $\Rightarrow$ ~~$[(U,\tilde{s})] = [(V,\tilde{s})] = j_x[(V,s)] = j_x(0)$~~

  $\Rightarrow$ 0 and $s|_V$ determine the same stalks $\forall x \in V$

  $\Rightarrow$ $0 = s|_V$ $\Rightarrow$ $[(U,s)] = [(V,0)] = 0$.

  $\Rightarrow$ j_x is injective.

- $[(U,\tilde{s})] \in \mathcal{F}^{\#}_x$ $\Rightarrow$ $\exists\, V$-open, $x \in V \subset U$, $\exists\, t \in \mathcal{F}(V)$, $\tilde{s}(x) = t_x$

  $\Rightarrow$ $[(U,s)] \simeq [(V,\tilde{t})] = j_x[(V,t)]$.

  $\Rightarrow$ j_x is surjective

§6 Kernels, Cokernels, Image :-

Definitions :-

① Subsheaf :- A sheaf $\mathcal{F}$ is a subsheaf of a sheaf $\mathcal{G}$ if

- $\forall U$-open, $\mathcal{F}(U)$ is a subgroup of $\mathcal{G}(U)$
- restriction maps for $\mathcal{F}$ are induced by the restriction maps for $\mathcal{G}$.

② Kernel :- Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves.

- The kernel of φ, is the presheaf $U \longmapsto \underbrace{\mathrm{Ker}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))}_{\ker(\varphi)(U)}$.
- Fact :- $\ker(\varphi)$ is a sheaf. (check this.)

A morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ of sheaves is injective if $\ker(\varphi) = 0$.
($\therefore \forall U$, $\varphi(U)$ is injective.)

③ ~~Image~~ Image :- Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves.

- The presheaf image of φ is $U \longmapsto \mathrm{Im}(\varphi(U))$
- The image of φ $\mathrm{im}(\varphi)$ is the sheafification of the presheaf image of φ.
  since at a presheaf level one has $\mathrm{Im}(\varphi(U)) \subset \mathcal{G}(U)$, one gets
  ~~natural~~ a map $\mathrm{im}(\varphi) \longrightarrow \mathcal{G}$ of sheaves
  Fact :- $\mathrm{im}(\varphi) \longrightarrow \mathcal{G}$ is injective. (Exercise!)
  So one can identify $\mathrm{im}(\varphi)$ as a subsheaf of $\mathcal{G}$.
- A morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ of sheaves is said to be surjective if $\mathrm{im}(\varphi) = \mathcal{G}$.

(4) Quotient :- Let $\mathcal{H}$ be a subsheaf of a sheaf $\mathcal{G}$.
The quotient $\mathcal{G}/\mathcal{H}$ is the sheafification of the presheaf
$$U \longmapsto \mathcal{G}(U)/\mathcal{H}(U).$$

(5) Cokernel :- Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of ~~presheaves~~ sheaves.
The cokernel of φ $\operatorname{coker}(\varphi)$ is the sheaf $\mathcal{G}/\operatorname{im}(\varphi)$.

Note:- Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves.

(i) φ is injective $\Leftrightarrow$ $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective $\forall U$.
$\Leftrightarrow$ $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ " " $\forall x \in X$.

($U \mapsto \ker(\varphi)(U)$ is already a sheaf !

(ii) φ is surjective $\Leftrightarrow$ $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is surjective $\forall x \in X$.
The maps $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ need not be surjective.

($U \longmapsto \operatorname{Im}(\varphi(U))$ is not a sheaf !) (

(6) A sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ of morphism of sheaves is exact if $\operatorname{Ker}(\psi) = \operatorname{Im}(\varphi)$.

Theorem

> The category S_X of sheaves (of abelian groups) on X is an abelian category.

§7 Changing the base space.

Let $f : X \longrightarrow Y$ be a continuous fn. of top. spaces.

We want to move sheaves on X, via f, to sheaves on Y

and also move sheaves on Y, " , " " X.

① **Direct image.** Let $\mathcal{F}$ be a sheaf on X. , $f : X \longrightarrow Y$ cont. fn.

The direct image $f_* \mathcal{F}$ is the sheaf on Y defined by:-

$$V \longmapsto f_*\mathcal{F}(V) := \mathcal{F}(f^{-1}(V))$$

(This is indeed a sheaf!) $\qquad f_* : S_X \longrightarrow S_Y$. is a functor.

② **Inverse Image.** :- Let $\mathcal{G}$ be a sheaf on Y. , $f : X \longrightarrow Y$ cont. fn.

The inverse image $f^*\mathcal{G}$ of $\mathcal{G}$ is the sheafification of the presheaf $f'\mathcal{G}$

$$U \longmapsto \varinjlim_{f(U) \subset V} \mathcal{G}(V) = f'\mathcal{G}(U) \qquad . \quad (f^*\mathcal{G} = f'\mathcal{G}^{\#})$$

This definition is made so that at the level of stalks we have:-

$$f^*\mathcal{G}_x = \mathcal{G}_{f(x)}.$$

$f^* : S_Y \longrightarrow S_X$ is a functor.

Theorem (Adjunction formula).

Let $f : X \longrightarrow Y$ be a continuous fn.

Then f^* is a left adjoint of f_* and f_* is a right adjoint of f^*,

i.e., $\forall$ sheaves $\mathcal{F}$ on X, and $\forall$ sheaves $\mathcal{G}$ on Y there is a

functorial isomorphism:-

$$\mathrm{Hom}_{S_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{S_Y}(\mathcal{G}, f_*\mathcal{F})$$

Proof :-

$$\mathrm{Hom}_{S_X}(f^*\mathcal{G}, \mathcal{F}) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_{S_Y}(\mathcal{G}, f_*\mathcal{F}).$$

α is defined as follows:-

Let $\varphi \in \mathrm{Hom}_{S_X}(f^*\mathcal{G}, \mathcal{F})$. Then $\varphi = \{\varphi_U\}$ with

$$\mathrm{Hom}_{S_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{PS_X}(f'\mathcal{G}, \mathcal{F})$$

$$\begin{tikzcd} \varphi_U : f^*\mathcal{G}(U) \arrow[r] & \mathcal{F}(U) \\ \varinjlim_{f(U)\subset V} \mathcal{G}(V) \arrow[u, equal] \arrow[ur, "\varphi_U"'] & \end{tikzcd}$$

$\alpha\varphi \in \mathrm{Hom}_{S_Y}(\mathcal{G}, f_*\mathcal{F})$. Let $V \underset{\text{open}}{\subset} Y$.

$$\alpha\varphi_V : \mathcal{G}(V) \longrightarrow f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

$$\begin{tikzcd} \mathcal{G}(V) \arrow[rr, "\alpha\varphi_V"] \arrow[dr] & & \mathcal{F}(f^{-1}(V)) \\ & \varinjlim_{f(f^{-1}(V))\subset W} \mathcal{G}(W) \arrow[ur, "\varphi_{f^{-1}(V)}"'] & \end{tikzcd}$$

β is defined as:

Let $\psi \in \mathrm{Hom}_{S_Y}(\mathcal{G}, f_*\mathcal{F})$ $\quad \psi = \{\psi_V\}$ $\quad \psi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}(V))$.

want $\beta\psi \in \mathrm{Hom}_{S_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{PS_X}(f'\mathcal{G}, \mathcal{F})$

want:- $\forall\, U \underset{\text{open}}{\subset} X$,

$$\begin{tikzcd} \beta\psi_U : f'\mathcal{G}(U) \arrow[r] & \mathcal{F}(U) \\ \varinjlim_{f(U)\subset V} \mathcal{G}(V) \arrow[u, equal] \arrow[ur] & \end{tikzcd}$$

To get a map from direct limit, it suffices to give a map from each piece:

want:- $f(U) \subset V$, $\mathcal{G}(V) \longrightarrow \mathcal{F}(U)$

$f(U) \subset V \Rightarrow$ $\quad U \subset f^{-1}(V)$.

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\psi_V"] \arrow[dr] & \mathcal{F}(f^{-1}(V)) \arrow[d, "r_{f^{-1}(V),U}"] \\ & \mathcal{F}(U) \end{tikzcd}$$

is the required map.

§8 Restrictions & Extensions by zero :-

Defn :- Let A be a subset of X.
Give A the subspace topology.
Let $i : A \hookrightarrow X$ be the inclusion map.
Given any sheaf $\mathcal{F}$ on X, the restriction of $\mathcal{F}$ to A, $\mathcal{F}|A$,
is defined as :- $\mathcal{F}|_A = i^*\mathcal{F}$,
the inverse image of $\mathcal{F}$ via i.

Note:- $\forall\ a \in A$, $(\mathcal{F}|_A)_a = \mathcal{F}_a$.

For extending a sheaf, outside a given subset, by zero is a little delicate. It depends on the "topology" of the subset.
If the subset is closed then simply take the direct image sheaf.
If the subset is open there is a possible problem at the boundary of U. (Look at homework problem #5; or Hartshorne Exercise I.