

Sheaf Cohomology as Derived Functors.

§1 Right derived functors of a covariant left exact functor.

Let \mathcal{A} be an abelian category.

An object $I \in \text{ob}(\mathcal{A})$ is said to be injective if $\text{Hom}_{\mathcal{A}}(-, I)$ is exact.

(This is the same as: - ~~given~~ dotted arrow exists. $\begin{matrix} 0 \rightarrow M \rightarrow N \\ \downarrow \swarrow \\ I \end{matrix}$)

We say \mathcal{A} has enough injectives if every object can be embedded inside an injective object.

Assume henceforth that \mathcal{A} has enough injectives, and for each object A in \mathcal{A} , find an injective resolution:

$$0 \rightarrow A \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \rightarrow \dots \rightarrow I^n \rightarrow \dots$$

Let $F: \mathcal{A} \rightarrow \mathcal{AB}$ be a covariant left exact functor.

Recall that the right derived functors are given by:

$$(R^i F)(A) = H^i(FI^0), \quad \forall i \geq 0$$

- $R^0 F = F$ (meaning naturally isomorphic)
- If I is injective then $R^i F(I) = 0 \quad \forall i \geq 1$. use the resolution $0 \rightarrow I \rightarrow I \rightarrow 0 \rightarrow 0$
- A short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ induces a long exact sequence $0 \rightarrow FA \rightarrow FB \rightarrow FC \xrightarrow{S^0} R^1 FA \rightarrow R^1 FB \rightarrow R^1 FC \xrightarrow{S^1} R^2 FA \rightarrow R^2 FB \rightarrow \dots$ and the connecting hom. S^i are natural in short exact sequences.

§2 Acyclic resolutions.

Defn:- Let \mathcal{A} be an abelian category with enough injectives.
Let $F: \mathcal{A} \rightarrow \mathcal{AB}$ be a covariant left exact functor.
An object $A \in \text{ob}(\mathcal{A})$ is said to be F-acyclic if

$$R^i F(A) = 0 \quad \forall i \geq 1.$$

Note:- I - injective $\Rightarrow I$ is F -acyclic.

Theorem (A simple principle of homological algebra)

Let \mathcal{A} and $F: \mathcal{A} \rightarrow \mathcal{AB}$ be as above.

Let $A \in \text{ob}(\mathcal{A})$

Let $0 \rightarrow A \rightarrow X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow \dots \rightarrow X^n \rightarrow \dots$

be an acyclic resolution for A , i.e., it is an exact sequence with each X^i being F -acyclic.

Then $R^i F(A) = H^i(FX^0)$

The point of this theorem is that once we have a construction of the derived functors $\{R^i F\}$ using injective resolutions, we can then forget the injective resolutions and ~~also~~ compute the functors using acyclic resolutions. In practice, an acyclic object ^{can be} is much more manageable than an injective object.

Proof:- Proof of this theorem is by induction on i .

$i=0$:- $0 \rightarrow A \rightarrow X^0 \rightarrow X^1$ is exact

$\Rightarrow 0 \rightarrow FA \rightarrow FX^0 \rightarrow FX^1$ is exact (F is left exact.)

$\Rightarrow FA = \text{Ker}(FX^0 \rightarrow FX^1) = H^0(FX^0)$

$\Rightarrow (R^0 F)(A) = H^0(FX^0)$.

$i=1$ Break up the acyclic resolution of A :-

$$0 \rightarrow A \rightarrow X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow X^3 \rightarrow \dots$$

as

$$0 \rightarrow A \rightarrow X^0 \rightarrow X^0/A \rightarrow 0$$

and

$$0 \rightarrow X^0/A \rightarrow X^1 \rightarrow X^2 \rightarrow X^3 \rightarrow \dots$$

• Now $0 \rightarrow A \rightarrow X^0 \rightarrow X^0/A \rightarrow 0$ is a short exact sequence which gives the long exact sequence :-

$$0 \rightarrow FA \rightarrow FX^0 \rightarrow F(X^0/A) \rightarrow R^1F(A) \rightarrow 0 \rightarrow R^1F(X^0/A) \rightarrow R^2F(A) \rightarrow 0 \rightarrow R^2F(X^0/A) \rightarrow \dots$$

• The exact sequence $0 \rightarrow X^0/A \rightarrow X^1 \rightarrow X^2 \rightarrow X^3 \rightarrow \dots$ is an acyclic resolution of X^0/A which we relabel as

$$0 \rightarrow X^0/A \rightarrow Y^0 \rightarrow Y^1 \rightarrow Y^2 \rightarrow \dots$$

$$R^1F(A) = \frac{F(X^0/A)}{\text{Ker}(F(X^0/A) \rightarrow R^1F(A))} = \frac{F(X^0/A)}{\text{Im}(FX^0 \rightarrow F(X^0/A))}$$

Now $0 \rightarrow X^0/A \rightarrow X^1 \rightarrow X^2$ is exact

$\Rightarrow 0 \rightarrow F(X^0/A) \rightarrow FX^1 \rightarrow FX^2$ is exact.

$$\Rightarrow F(X^0/A) = \text{Ker}(FX^1 \rightarrow FX^2).$$

$$\text{Im}(FX^0 \rightarrow F(X^0/A)) = \text{Im}(FX^0 \rightarrow FX^1).$$

$$\begin{aligned} \Rightarrow R^1F(A) &= \frac{F(X^0/A)}{\text{Im}(FX^0 \rightarrow F(X^0/A))} = \frac{\text{Ker}(FX^1 \rightarrow FX^2)}{\text{Im}(FX^0 \rightarrow FX^1)} \\ &= H^1(FX^0) \end{aligned}$$

This proves the $i=1$ case.

For $i \geq 2$:-

Assume that for all $0 \leq i \leq n$ we have checked that $R^i F(A)$ is computed as $H^i(FX^0)$ where $0 \rightarrow A \rightarrow X^0 \rightarrow X^1 \rightarrow \dots$ is any acyclic resolution of A .

For $i = n+1$:-

$$R^{n+1} F(A) = R^n F(X^0/A) \simeq H^n(FY^0) = H^{n+1}(FX^0)$$

Long exact sequence

Induction

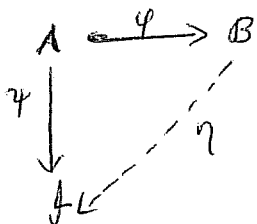
Relabelling of
as

$$\begin{array}{l} X^1 \rightarrow X^2 \rightarrow X^3 \rightarrow \dots \\ Y^0 \rightarrow Y^1 \rightarrow Y^2 \rightarrow \dots \end{array}$$

§3 Injective Sheaves.

Let X be a top. space.
Consider sheaves of abelian groups on X .

Defn:- A sheaf \mathcal{F} on X is said to be an injective sheaf if
in any diagram



of sheaves & morphisms

with $\text{Ker}(\varphi) \subset \text{Ker}(\eta)$, one can find a morphism $\eta: B \rightarrow \mathcal{F}$ such that $\varphi = \eta \circ \varphi$.

Theorem

The category S_X of sheaves of abelian groups on X has enough injectives.

Proof (Godement).

Let \mathcal{F} be a sheaf on X .

$\forall x \in X$, let I_x be an injective abelian group (\Leftrightarrow divisible ab. group.)
such that $\mathcal{F}_x \hookrightarrow I_x$.
(Every abelian group is a subgroup of an injective ab. group.)

Define $\mathcal{I} := \dots$
 $u \subset \text{span } X$,
 $v \subset u$,

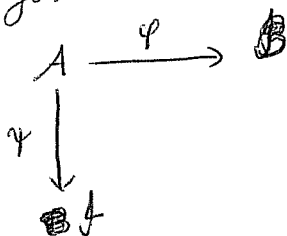
$$\mathcal{I}(u) = \prod_{x \in u} I_x.$$

Restriction map: $\prod_{x \in u} I_x \longrightarrow \prod_{x \in v} I_x$
is simply projection.

check:- (i) \mathcal{I} is an injective sheaf.
(ii) $\mathcal{F} \hookrightarrow \mathcal{I}$.

To check \mathcal{I} is injective just do it at the level of stalks!

Say, we are given



with $\text{Ker}(\varphi) \subset \text{Ker}(\eta)$.

Then we have a diagram of abelian groups.

$$\begin{array}{ccc} A_x & \xrightarrow{\varphi_x} & B_x \\ \varphi_x \downarrow & & \\ A_x & & \end{array}$$

Since A_x is injective as an ab. group, we get

$$\begin{array}{ccc} A_x & \xrightarrow{\varphi_x} & B_x \\ \varphi_x \downarrow & \swarrow \eta_x & \\ A_x & & \end{array}$$

Put the η_x 's together to get $\eta: B \rightarrow \mathcal{F}$ as $\eta_U: B(U) \xrightarrow{\Delta_U} \mathcal{F}(U)$

To see $\mathcal{F} \hookrightarrow \mathcal{F}$, we need an embedding $\mathcal{F}(U) \hookrightarrow \mathcal{F}(U)$.

$$\begin{aligned} \text{This is given by } \mathcal{F}(U) &\longrightarrow F_x \quad \forall x \in U \\ \Rightarrow \mathcal{F}(U) &\longrightarrow \prod_{x \in U} F_x \subset \prod_{x \in U} A_x = \mathcal{F}(U). \end{aligned}$$

Cor

Every sheaf \mathcal{F} has an injective resolution:-

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \mathcal{F}^2 \rightarrow \dots \rightarrow \mathcal{F}^n \rightarrow \dots$$

Definition of Sheaf cohomology.

Let \mathcal{F} be a sheaf of abelian groups on X .

$$H^i(X, \mathcal{F}) = H^i(\mathcal{F}^*(X)).$$

Note:- The resolution $0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots \rightarrow \mathcal{F}^n \rightarrow \dots$ gives a complex upon applying the global sections functor:

$$\mathcal{F}^0(X) \rightarrow \mathcal{F}^1(X) \rightarrow \mathcal{F}^2(X) \rightarrow \dots \rightarrow \mathcal{F}^n(X) \rightarrow \dots$$

§4 Flabby Sheaves.

Defn:- A sheaf \mathcal{F} on a space X is called a flabby sheaf if for any open set $U \subset X$, the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is surjective.

(One may equivalently define flabby sheaf by asking $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ to be surjective \forall open sets U, V with $V \subset U$.)

Note:- A flabby sheaf is also called a flasque sheaf.

(Flasque is a French word that translates to flabby in English.)

Nonexample:-

① The constant sheaf A_X determined by an abelian group on a space X is, in general, not a flabby sheaf.

(say U -connected, V -disconnected, open subset of U --- etc.)

Theorem (Basic properties of flabby sheaves).

① Injective \Rightarrow Flabby.

② Flabby \Rightarrow Acyclic.

③ Every sheaf can be embedded into a flabby sheaf.

④ Every sheaf \mathcal{F} has a resolution by flabby sheaves; ~~and~~ this resolution can be used to compute $H^i(X, \mathcal{F})$.

Proof:- ① Let \mathcal{F} be an injective sheaf.

Let $U \subset_{\text{open}} X$. (want: $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is surjective.)

Let $j: U \hookrightarrow X$ be the inclusion map.

~~Let~~ $\mathcal{F}|_U$ is the rest. to $(\mathcal{F}|_U)(w) = \mathcal{F}(w)$
 $\forall w \in U$

Let $Z = X - U \subset_{\text{closed}} X$ and

$i: Z \hookrightarrow X$ be the inclusion map.

Recall (from Ex. 5 on HW 5) that we have an exact sequence:-

$$0 \rightarrow j_!(\mathcal{F}|_U) \rightarrow \mathcal{F} \rightarrow j_*(\mathcal{F}|_Z) \rightarrow 0.$$

where:-

$j_!(\mathcal{F}|_U)$ is the sheaf associated to the presheaf:-

$$v \in \text{open } X \text{ :-}$$

$$v \mapsto (\mathcal{F}|_U)(v) \quad \text{if } v \subset U$$

$$v \mapsto 0 \quad \text{if } v \not\subset U.$$

$$\Rightarrow v\text{-fun } \subset X, \quad v \mapsto \begin{cases} \mathcal{F}(v), & v \subset U \\ 0, & v \not\subset U \end{cases}$$

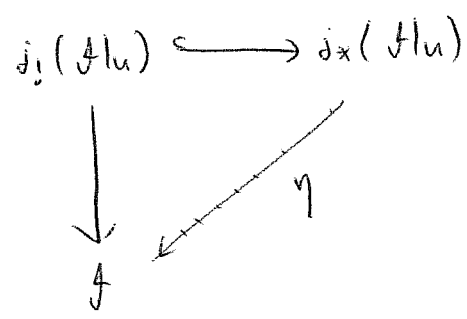
clearly, we have an injective morphism:-

$$j_!(\mathcal{F}|_U) \rightarrow j_*(\mathcal{F}|_U).$$

$$j_*(\mathcal{F}|_U)(v) = \mathcal{F}|_U(u \cap v) = \mathcal{F}(u \cap v).$$

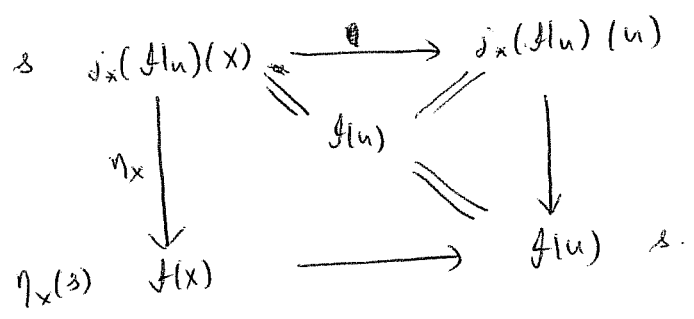
$v \in \text{open } X$

Look at the diagram



to get η by injectivity of \mathcal{F} .

Let $s \in \mathcal{F}(U)$. Then $s \in j_*(\mathcal{F}|_U)(X) = \mathcal{F}(U) = (j_*(\mathcal{F}|_U))(U)$.
 Let $\eta_x(s) = t \in \mathcal{F}(X)$.



2nd proof of ① :-

(Assuming ③) - which has a totally independent proof by Grothendieck of embedding \mathcal{F} into its sheaf of discontinuous sections, which is a flabby sheaf.)

~~One does~~ Let \mathcal{F} be an injective sheaf. Embed \mathcal{F} into a flabby sheaf \mathcal{F} .

$$\mathcal{F} \xrightarrow{i} \mathcal{F}$$

Now check that an injective subsheaf of a flabby sheaf is flabby:-

Since \mathcal{F} is injective we have a morphism $\eta: \mathcal{F} \rightarrow \mathcal{A}$ s.t.:-

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & \mathcal{F} \\ & & \searrow \eta \\ & & \mathcal{A} \end{array}$$

Let U be open in X , one has:-

$$\begin{array}{ccc} \mathcal{F}(U) & \hookrightarrow & \mathcal{F}(U) \\ & & \searrow \eta_U \\ & & \mathcal{A}(U) \end{array} \Rightarrow \eta_U \text{ is surjective}$$

Now look at the diagram:-

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \end{array} \Rightarrow \mathcal{F}(X) \longrightarrow \mathcal{A}(U) \Rightarrow \mathcal{F} \text{ is flabby.}$$

Proof of ②:- Flabby \Rightarrow Acyclic.

Let \mathcal{F} be a flabby sheaf.

Since S_X has enough injectives, embed $\mathcal{F} \hookrightarrow \mathcal{A}$ with \mathcal{A} -injective.

Consider the exact sequence of sheaves.

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{A} \rightarrow \mathcal{G} \rightarrow 0$$

where $\mathcal{G} = \mathcal{A}/\mathcal{F}$.

~~we have seen that \mathcal{A} being injective \mathcal{A} is flabby.~~

~~we have seen that \mathcal{A} being injective \mathcal{A} is flabby.~~

Take the long exact sequence in sheaf cohomology attached to $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{G} \rightarrow 0$:-

$$0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{I}(X) \rightarrow \mathcal{G}(X) \rightarrow H^1(X, \mathcal{F}) \rightarrow H^1(X, \mathcal{I}) \rightarrow H^1(X, \mathcal{G}) \rightarrow H^2(X, \mathcal{F}) \rightarrow \dots$$

\parallel
 0

Observe the following:-

• \mathcal{I} -injective $\Rightarrow \mathcal{I}$ -acyclic $\Rightarrow H^i(X, \mathcal{I}) = 0 \quad \forall i \geq 1$.

• By defn, $0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{I}(X) \rightarrow \mathcal{G}(X) \rightarrow 0$ is exact

$$\Rightarrow H^1(X, \mathcal{F}) = 0.$$

That is $H^1(X, \mathcal{F}) = 0$ for any flabby sheaf \mathcal{F} .

• Since $H^n(X, \mathcal{I}) = 0 \quad \forall n \geq 1$, $H^n(X, \mathcal{G}) \cong H^{n+1}(X, \mathcal{F})$

\mathcal{F} -flabby
 \mathcal{I} -injective $\Rightarrow \mathcal{I}$ -flabby $\} \Rightarrow \mathcal{G}$ -flabby.

By induction $H^n(X, \mathcal{G}) = 0 \Rightarrow H^{n+1}(X, \mathcal{F}) = 0$.

— This completes the proof of ②.

Pf of ③:- Every sheaf can be embedded into a flabby sheaf.

Let \mathcal{F} be a sheaf.

Consider the sheaf $\mathcal{S}(\mathcal{F})$ of discontinuous sections of \mathcal{F} :-

$$\mathcal{S}(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x \quad \left| \quad \mathcal{S}(x) \in \mathcal{F}_x \right.$$

• $\mathcal{S}(\mathcal{F})(U) \rightarrow \mathcal{S}(\mathcal{F})(V)$ is simply restriction of the discont. sec. $\mathcal{S}|_V$.

There is an obvious embedding $\mathcal{F} \hookrightarrow \mathcal{S}(\mathcal{F})$.

$$\mathcal{F}(U) \rightarrow \mathcal{S}(\mathcal{F})(U)$$

$$s \mapsto \tilde{s}$$

• $\tilde{s}(x)$ = germ at x determined by (U, s) .

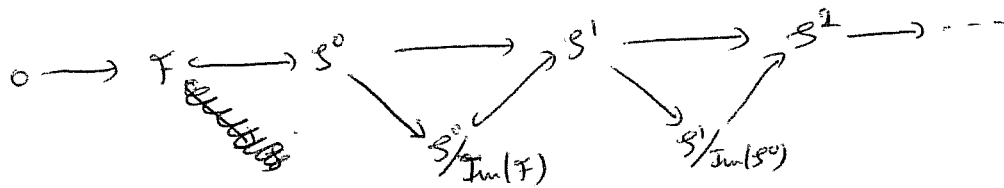
• $s \mapsto \tilde{s}$ is injective :- $\tilde{s} = 0 \Leftrightarrow \tilde{s}(x) = 0 \quad \forall x \in U$
 $\Leftrightarrow s$ is locally 0 on U .

$S(F)$ - sometimes called the Godement flabby envelope of F .

Observe that $S(F)$ is indeed flabby: $S(F)(U) \longrightarrow S(F)(V)$.

If $\tilde{S} \in S(F)(V)$, then just extend \tilde{S} by 0 outside of V .

Pf of (2) :- This follows from (3) by the "usual" argument :-



This is called Godement's canonical flabby resolution of F .

The first Čech cohomology group:

This motivates the definition of Čech cohomology groups exactly like the calculation which motivated the definition of group cohomology.

The question is:- What is the obstruction to the exactness of the global sections functor.

Let $0 \rightarrow F \rightarrow G \rightarrow \mathcal{A} \rightarrow 0$ be a short exact sequence of sheaves. Then upon taking global sections,

$$0 \rightarrow F(x) \rightarrow G(x) \rightarrow \mathcal{A}(x) \text{ is exact.}$$

In general, $G(x) \rightarrow \mathcal{A}(x)$ is not surjective. Let us see how this fails:-

Let $s \in \mathcal{A}(x)$.

Since $G \rightarrow \mathcal{A}$, s can be locally lifted, i.e., \exists open cover $\mathcal{U} = \{U_i\}$ of X ($x = \bigcup_i U_i$), $\exists s_i \in G(U_i)$ s.t. $s_i|_{U_i} \rightarrow s|_{U_i}$ in $\mathcal{A}(U_i)$.

Consider, $\# i, j$, $s_{ij} = s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in G(U_i \cap U_j)$

Clearly, $s_{ij}|_{U_i \cap U_j} \rightarrow 0$ in $\mathcal{A}(U_i \cap U_j)$

$$\Rightarrow \exists t_{ij} \in F(U_i \cap U_j) \text{ s.t.}$$

$$t_{ij}|_{U_i \cap U_j} \rightarrow s_{ij} = s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}$$

(since $0 \rightarrow F(U) \rightarrow G(U) \rightarrow \mathcal{A}(U) \rightarrow 0$ is exact $\forall U \subseteq_{\text{open}} X$)

So, ~~given~~ ^{for the} open cover \mathcal{U} , we have $\{t_{ij}\}$ which satisfy the cocycle relation:-

$$t_{ij} + t_{jk} + t_{ki} = 0 \text{ on } U_i \cap U_j \cap U_k$$

or

$$t_{ij} - t_{ik} + t_{jk} = 0 \text{ on } U_i \cap U_j \cap U_k$$

Defn:- The 1-cocycles for the sheaf \mathcal{F} ~~for~~ with respect to the covering \mathcal{U} is defined as

$$Z^1(X, \mathcal{U}, \mathcal{F}) = \left\{ (t_{ij})_{i,j \in I \times I} \mid \begin{array}{l} 1. t_{ij} \in \mathcal{F}(U_i \cap U_j) \\ 2. t_{ij} - t_{ik} + t_{jk} = 0 \text{ on } U_i \cap U_j \cap U_k \end{array} \right\}$$

Defn:- The 1-coboundaries are:-

$$B^1(X, \mathcal{U}, \mathcal{F}) = \left\{ (t_{ij}) \in Z^1(X, \mathcal{U}, \mathcal{F}) \mid \exists s_i \in \mathcal{F}(U_i) \text{ for which } t_{ij} = s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \right\}$$

Defn:- $H^1(X, \mathcal{U}, \mathcal{F}) = Z^1(X, \mathcal{U}, \mathcal{F}) / B^1(X, \mathcal{U}, \mathcal{F})$

$\check{H}^1(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^1(X, \mathcal{U}, \mathcal{F})$

Direct limit over all coverings \mathcal{U} , $\mathcal{U}_1 \subseteq \mathcal{U}_2$ if \mathcal{U}_2 is a refinement of \mathcal{U}_1 .

$\check{H}^1(X, \mathcal{F}) =$ first Čech cohomology group of X with coeffs. in \mathcal{F} .

We are looking at an "exact" sequence:-

$$0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{G}(X) \rightarrow \mathcal{H}(X) \rightarrow \check{H}^1(X, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{G}) \rightarrow \check{H}^1(X, \mathcal{H})$$

(Later we will define

• Čech cohomology groups $\check{H}^q(X, \mathcal{U}, \mathcal{F})$

• if X is reasonably nice, and \mathcal{U} is fine enough then. $\check{H}^q(X, \mathcal{U}, \mathcal{F}) = H^q(X, \mathcal{F})$