Lecture-15 & 16

Sheaf Cohomology as Derived Functors.

§1 Right derived functors of a covariant left exact functor.

Let $\mathcal{A}$ be an abelian category.

An object $I \in ob(\mathcal{A})$ is said to be *injective* if $Hom_{\mathcal{A}}(-, I)$ is exact.

(This is the same as:- ~~Given~~ Dotted arrow exists.
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[dl, dashed] \\ & I & \end{tikzcd}$$
)

We say $\mathcal{A}$ has *enough injectives* if every object can be embedded inside an injective object.

Assume henceforth that $\mathcal{A}$ has enough injectives, and for each object A in $\mathcal{A}$, fix an injective resolution:-

$$0 \to A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^1 \to \cdots \to I^n \to \cdots$$

Let $F : \mathcal{A} \to AB$ be a *covariant left exact functor.*

Recall that the *right derived functors* are given by:-

$$\boxed{(R^iF)(A) = H^i(FI^\circ), \quad \forall i \geq 0.}$$

- $R^0F = F$ (meaning naturally isomorphic)
- If I is injective then $R^iF(I) = 0 \quad \forall i \geq 1$. use the resolution $0 \to I \to I \to 0 \to 0$
- A short exact sequence $0 \to A \to B \to C \to 0$ induces a long exact sequence

$$0 \to FA \to FB \to FC \xrightarrow{\delta^0} R^1FA \to R^1FB \to R^1FC \xrightarrow{\delta^1} R^2FA \to R^2FB \to \cdots$$

and the connecting hom. δ° are natural in short exact sequences.

§2 Acyclic resolutions.

Defn:- Let $\mathcal{A}$ be an abelian category with enough injectives.
Let $F : \mathcal{A} \longrightarrow AB$ be a covariant left exact functor.
An object $A \in ob(\mathcal{A})$ is said to be F-acyclic if

$$R^i F(A) = 0 \quad \forall i \geq 1.$$

Note:- I - injective $\Rightarrow$ I is F-acyclic.

Theorem (A simple principle of homological algebra)

Let $\mathcal{A}$ and $F : \mathcal{A} \longrightarrow AB$ be as above.
Let $A \in ob(\mathcal{A})$
Let $0 \longrightarrow A \to X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots \longrightarrow X^n \longrightarrow \cdots$
be an acyclic resolution for A., i.e., it is an exact sequence with each X^i being F-acyclic.

Then
$$R^i F(A) = H^i(FX^\bullet)$$

The point of this theorem is that once we have a construction of the derived functors $\{R^i F\}$ using injective resolutions, we can then forget the injective resolutions and ~~also~~ compute the functors using acyclic resolutions. In practice, an acyclic object can be much more manageable than an injective object.

Proof:- Proof of this theorem is by induction on i.

$i=0$:- $0 \longrightarrow A \longrightarrow X^0 \longrightarrow X^1$ is exact

$\Rightarrow$ $0 \longrightarrow FA \longrightarrow FX^0 \longrightarrow FX^1$ is exact (F is left exact.)

$\Rightarrow$ $FA = \operatorname{Ker}(FX^0 \longrightarrow FX^1) = H^0(FX^\bullet)$

$\Rightarrow$ $(R^0F)(A) = H^0(FX^\bullet)$.

$\underline{\underline{i=1}}$ Break up the acyclic resolution of A :-

$$0 \to A \to X^0 \to X^1 \to X^2 \to X^3 \to \cdots$$

as

$$0 \to A \to X^0 \to X^0/A \to 0$$

and

$$0 \to X^0/A \to X^1 \to X^2 \to X^3 \to \cdots.$$

- Now $0 \to A \to X^0 \to X^0/A \to 0$ is a short exact sequence which gives the long exact sequence :-

$$0 \to FA \to FX^0 \to F(X^0/A) \to R^1F(A) \to 0 \to R^1F(X^0/A) \to R^2F(A) \to 0 \to R^2F(X^0/A) \to \cdots$$

- The exact sequence $0 \to X^0/A \to X^1 \to X^2 \to X^3 \to \cdots$ is an acyclic resolution of X^0/A which we relabel as

$$0 \to X^0/A \to Y^0 \to Y^1 \to Y^2 \to \cdots$$

- $$R^1F(A) = \frac{F(X^0/A)}{\operatorname{Ker}(F(X^0/A) \to R^1F(A))} = \frac{F(X^0/A)}{\operatorname{Im}(FX^0 \to F(X^0/A))}$$

Now $0 \to X^0/A \to X^1 \to X^2$ is exact

$\Rightarrow$ $0 \to F(X^0/A) \to FX^1 \to FX^2$ is exact.

$\Rightarrow$. $F(X^0/A) = \operatorname{Ker}(FX^1 \to FX^2)$.

. $\operatorname{Im}(FX^0 \to F(X^0/A)) = \operatorname{Im}(FX^0 \to FX^1)$.

$$\Rightarrow \quad R^1F(A) = \frac{F(X^0/A)}{\operatorname{Im}(FX^0 \to F(X^0/A))} = \frac{\operatorname{Ker}(FX^1 \to FX^2)}{\operatorname{Im}(FX^0 \to FX^1)}$$

$$= H^1(FX^\bullet)$$

This proves the $i=1$ case.

For $i \geq 2$:-

Assume that for all $0 \leq i \leq n$ we have checked that $R^i F(A)$ is computed as $H^i(FX^\bullet)$ where $0 \to A \to X^0 \to X^1 \to \cdots$ is <u>any</u> acyclic resolution of A.

For $i = n+1$:-

$$R^{n+1}F(A) \underset{\text{Long exact sequence}}{=} R^n F(X^0/A) \underset{\text{Induction}}{\simeq} H^n(FY^\bullet) \underset{\text{Relabelling of } X^1 \to X^2 \to X^3 \to \cdots \text{ as } Y^0 \to Y^1 \to Y^2 \to \cdots}{=} H^{n+1}(FX^\bullet)$$

—.

§3 Injective Sheaves.

Let X be a top. space.
Consider sheaves of abelian groups on X.

Defn:- A sheaf $\mathcal{I}$ on X is said to be an <u>injective sheaf</u> if in any diagram

$$\begin{tikzcd} A \arrow[r, hook, "\varphi"] \arrow[d, "\psi"'] & B \arrow[dl, dashed, "\eta"] \\ \mathcal{I} & \end{tikzcd}$$

of sheaves & morphisms with $\mathrm{Ker}(\varphi) \subset \mathrm{Ker}(\psi)$, one can find a morphism $\eta : B \to \mathcal{I}$ such that $\psi = \eta \circ \varphi$.

Theorem

$$\boxed{\text{The category } S_X \text{ of sheaves of abelian groups on } X \text{ has enough injectives.}}$$

Proof (Godement).

Let $\mathcal{F}$ be a sheaf on X.

$\forall\, x \in X$, let I_x be an injective abelian group ($\Leftrightarrow$ divisible ab. group) such that $\mathcal{F}_x \hookrightarrow I_x$. (Every abelian group is a subgroup of an injective ab. group.)

Define $\mathcal{I}$:- $U \underset{\text{open}}{\subset} X$, $\quad \mathcal{I}(U) = \prod_{x \in U} I_x$.

$V \subset U$, Restriction map : $\prod_{x \in U} I_x \longrightarrow \prod_{x \in V} I_x$ is simply projection.

Check:- (i) $\mathcal{I}$ is an injective sheaf.

(ii) $\mathcal{F} \hookrightarrow \mathcal{I}$.

To check $\mathcal{I}$ is injective just do it at the level of stalks!

Say, we are given

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\psi"'] & B \\ \mathcal{I} & \end{tikzcd}$$

with $\mathrm{Ker}(\varphi) \subset \mathrm{Ker}(\psi)$.

Then we have a diagram of abelian groups.

$$\begin{tikzcd} A_x \arrow[r, "\varphi_x"] \arrow[d, "\psi_x"'] & B_x \\ I_x & \end{tikzcd}$$

Since I_x is injective as an ab. group, we get

$$\begin{tikzcd} A_x \arrow[r, "\varphi_x"] \arrow[d, "\psi_x"'] & B_x \arrow[dl, "\eta_x"] \\ I_x & \end{tikzcd}$$

Put the η_x's together to get $\eta : \mathcal{B} \longrightarrow \mathcal{I}$ as $\eta_U : \mathcal{B}(U) \longrightarrow \mathcal{I}(U)$, $s \longmapsto ([U,s]_x)$

To see $\mathcal{F} \hookrightarrow \mathcal{I}$, we need an embedding $\mathcal{F}(U) \hookrightarrow \mathcal{I}(U)$.

This is given by $\mathcal{F}(U) \longrightarrow F_x \quad \forall x \in U$

$$\Rightarrow \quad \mathcal{F}(U) \longrightarrow \prod_{x \in U} \mathcal{F}_x \subset \prod_{x \in U} I_x = \mathcal{I}(U).$$

Cor:

$$\boxed{\begin{array}{l}\text{Every sheaf } \mathcal{F} \text{ has an injective resolution :-} \\ 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots \longrightarrow \mathcal{I}^n \longrightarrow \cdots\end{array}}$$

Definition of Sheaf cohomology.

$$\boxed{\begin{array}{l}\text{Let } \mathcal{F} \text{ be a sheaf of abelian groups on } X. \\ H^i(X, \mathcal{F}) = H^i(\mathcal{I}^\bullet(X)).\end{array}}$$

Note:- The resolution $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots \longrightarrow \mathcal{I}^n \longrightarrow \cdots$

gives a complex upon applying the global sections functor:

$$\mathcal{I}^0(X) \longrightarrow \mathcal{I}^1(X) \longrightarrow \mathcal{I}^2(X) \longrightarrow \cdots \longrightarrow \mathcal{I}^n(X) \longrightarrow \cdots$$

§4 Flabby Sheaves.

Defn:- A sheaf $\mathcal{F}$ on a space X is called a flabby sheaf if for any open set $U \subset X$, the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(U)$ is surjective.

(One may equivalently define flabby sheaf by asking $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ be surjective $\forall$ open sets U, V with $V \subset U$.)

Note:- A flabby sheaf is also called a flasque sheaf.

(Flasque is a French word that translates to flabby in English.)

Nonexamples:-

① The constant sheaf A_X determined by an abelian group on a space X is, in general, not a flabby sheaf.

(Say U - connected, V - disconnected, open subset of U ... etc.)

Theorem (Basic properties of flabby sheaves)

① Injective $\Rightarrow$ Flabby.

② Flabby $\Rightarrow$ Acyclic.

③ Every sheaf ~~can~~ be embedded into a flabby sheaf.

④ Every sheaf $\mathcal{F}$ has a resolution by flabby sheaves; this resolution can be used to compute $H^i(X, \mathcal{F})$.

Proof:- ① Let $\mathcal{I}$ be an injective sheaf.

(want: $\mathcal{I}(X) \longrightarrow \mathcal{I}(U)$ is surjective.)

Let $U \underset{\text{open}}{\subset} X$.

Let $j : U \hookrightarrow X$ be the inclusion map.

$\mathcal{I}|_U$ is the rest. to
$(\mathcal{I}|_U)(W) = \mathcal{I}(W)$
$\forall W \underset{\text{open}}{\subset}$

Let $Z = X - U \underset{\text{closed}}{\subset} X$ and

$i : Z \hookrightarrow X$ be the inclusion map.

Recall (from Ex. 5 on HW 5) that we have an exact sequence:-

$$0 \longrightarrow j_!(I|_U) \longrightarrow I \longrightarrow i_*(I|_Z) \longrightarrow 0.$$

where:- $V \underset{\text{open}}{\subseteq} X$:-

$j_!(I|_U)$ is the sheaf associated to the presheaf:- $V \longmapsto (I|_U)(V)$ if $V \subset U$; $V \longmapsto 0$ if $V \not\subset U$.

$\Rightarrow$ V-open $\subset X$, $V \longmapsto \begin{cases} I(V), & V \subset U \\ 0, & V \not\subset U \end{cases}$

Clearly, we have an injective morphism:-

$$j_!(I|_U) \longrightarrow j_*(I|_U).$$

$j_*(I|_U)(V) = I|_U(U \cap V) = I(U \cap V)$. $\forall V \underset{\text{open}}{\subseteq}$

Look at the diagram

$$\begin{tikzcd} j_!(I|_U) \arrow[r, hook] \arrow[d] & j_*(I|_U) \arrow[dl, dashed, "\eta"] \\ I & \end{tikzcd}$$

to get η by injectivity of I.

Let $s \in I(U)$. Then $s \in j_*(I|_U)(X) = I(U) = (j_*(I|_U))(U)$.

Let $\eta_X(s) = t \in I(X)$.

$$\begin{tikzcd} s & j_*(I|_U)(X) \arrow[r] \arrow[d, "\eta_X"] \arrow[dr, phantom, "I(U)"] & j_*(I|_U)(U) \arrow[d] \\ \eta_X(s) & I(X) \arrow[r] & I(U) & s. \end{tikzcd}$$

<u>2nd proof of ①</u>:-

(Assuming ③ - which has a totally independent proof by Godement of embedding F into its sheaf of discontinuous sections. which is a flabby sheaf.)

~~One sheaf~~ Let I be an injective sheaf. $I \overset{i}{\hookrightarrow} F$

Embed I into a flabby sheaf F.

Now check that an injective subsheaf of a flabby sheaf is flabby:-

Since $\mathcal{I}$ is injective we have a morphism $\eta : \mathcal{F} \longrightarrow \mathcal{I}$ s.t.:-

$$\begin{tikzcd}
\mathcal{I} \arrow[r, hook, "i"] \arrow[d, equal] & \mathcal{F} \arrow[dl, "\eta"] \\
\mathcal{I} &
\end{tikzcd}$$

Let U be open in X, one has:-

$$\begin{tikzcd}
\mathcal{I}(U) \arrow[r, hook] \arrow[d, equal] & \mathcal{F}(U) \arrow[dl, "\eta_U"] \\
\mathcal{I}(U) &
\end{tikzcd}$$

$\Rightarrow$ η_U is surjec

Now look at the diagram:-

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, two heads] \arrow[d, two heads] & \mathcal{I}(X) \arrow[d] \\
\mathcal{F}(U) \arrow[r, two heads] & \mathcal{I}(U)
\end{tikzcd}$$

$\Rightarrow$ $\mathcal{I}(X) \twoheadrightarrow \mathcal{I}(U)$

$\Rightarrow$ $\mathcal{I}$ is flabby.

<u>Proof of ②</u>:- Flabby $\Rightarrow$ Acyclic.

Let $\mathcal{F}$ be a flabby sheaf.

Since S_X has enough injectives, embed $\mathcal{F} \hookrightarrow \mathcal{I}$ with $\mathcal{I}$ - injecti

Consider the exact sequence of sheaves. where $\mathcal{G} = \mathcal{I}/\mathcal{F}$.

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

Claim:- If $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$ is an exact sequence of sheaves, and $\mathcal{F}$ is a flabby then the sequence of global sections:-

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\alpha_X} \mathcal{H}(X) \xrightarrow{\beta_X} \mathcal{G}(X) \longrightarrow 0$$

is exact.

Pf of claim:- Taking global sections is left exact. So it is only surjectivity of $\mathcal{H}(X) \longrightarrow \mathcal{G}(X)$ that must be checked. Let $s \in \mathcal{G}(X)$. Since $\mathcal{H} \twoheadrightarrow \mathcal{G}$, s can ~~only~~ be locally lifted. Now consider the collection $\mathcal{C} = \{ (U, t) \mid U \subseteq_{\text{open}} X,\ t \in \mathcal{H}(U) \text{ s.t } t \mapsto s|_U \}$.

Since s can locally lifted, $\mathcal{C}$ is not empty. Put a partial order on $\mathcal{C}$ as $(U, t) \leq (U', t')$ if $U \subset U'$ and $t'|_U = t$. If there is a chain $\{(U_\alpha, t_\alpha)\}$ in $\mathcal{C}$ then it has a max. element in $\mathcal{C}$. (Why?)

Let (U_0, t_0) be a maximal element in $\mathcal{C}$. (by Zorn's lemma). If $U_0 = X$ then we are done. If $U_0 \subsetneq X$, then let $x \in X - U_0$, let V be a nhbd. of x, small enough such that $s|_V$ has a lift, i.e, $\exists\, t_V \in \mathcal{H}(V)$ s.t $t_V \longmapsto s|_V$. Consider $t_0 - t_V$ on $U \cap V$. This maps to 0 in $\mathcal{G}(U \cap V)$, because both t_0 & t_V map to $s|_{U \cap V}$, i.e, $\exists\, w \in \mathcal{F}(U \cap V)$ s.t $w \longmapsto t_0 - t_V \in \mathcal{H}(U \cap V)$.

But $\mathcal{F}$ is flabby $\Rightarrow \mathcal{F}(V) \twoheadrightarrow \mathcal{F}(U \cap V) \Rightarrow \exists\, w_V \in \mathcal{F}(V)$ which maps to w. Consider $t_V + \alpha_V(w_V) \in \mathcal{H}(V)$. Then $t_0 \in \mathcal{H}(U_0)$ & $t_V + \alpha_V(w_V) \in \mathcal{H}(V)$ patch up on $U \cap V$. $\Rightarrow \exists\, t \in \mathcal{H}(U_0 \cup V)$ s.t $t \longmapsto s|_{U_0 \cup V}$.

This contradicts maximality of (U_0, t_0). Hence $U_0 = X$.

This proves the claim.

The next point to observe is that:-

If $0 \to \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow \mathcal{G} \longrightarrow 0$ is an exact sequence & $\mathcal{F}, \mathcal{H}$ are flabby then $\mathcal{G}$ is flabby.

Take the long exact sequence in sheaf cohomology attached to $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$:-

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{G}(X) \to H^1(X,\mathcal{F}) \to \underset{0}{\underset{\|}{H^1(X,\mathcal{I})}} \to H^1(X,\mathcal{G}) \to H^2(X,\mathcal{F}) \to \cdots$$

Observe the following :-

- $\mathcal{I}$ - injective $\Rightarrow$ $\mathcal{I}$ - acyclic $\Rightarrow$ $H^i(X,\mathcal{I}) = 0 \quad \forall i \geq 1.$
- By claim, $0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{G}(X) \to 0$ is exact

$\Rightarrow \quad H^1(X,\mathcal{F}) = 0.$

That is $H^1(X,\mathcal{F}) = 0$ for <u>any</u> flabby sheaf $\mathcal{F}$.

- Since $H^n(X,\mathcal{I}) = 0 \ \forall n \geq 1$, $\quad H^n(X,\mathcal{G}) \cong H^{n+1}(X,\mathcal{F})$

$$\left.\begin{array}{l} \mathcal{F}\text{ - flabby} \\ \mathcal{I}\text{ - injective} \Rightarrow \mathcal{I}\text{ - flabby} \end{array}\right\} \Rightarrow \mathcal{G}\text{ - flabby}.$$

By induction $H^n(X,\mathcal{G}) = 0 \quad \Rightarrow H^{n+1}(X,\mathcal{F}) = 0.$

— This completes the proof of ②.

<u>Pf of ③</u> :- Every sheaf can be embedded into a flabby sheaf.

Let $\mathcal{F}$ be a sheaf.

Consider the <u>sheaf $\mathcal{S}(\mathcal{F})$ of discontinuous sections</u> of $\mathcal{F}$:-

- $\mathcal{S}(\mathcal{F})(U) = \{ \tilde{s} : U \to \prod_{x \in U} \mathcal{F}_x \mid \tilde{s}(x) \in \mathcal{F}_x . \}$
- $\mathcal{S}(\mathcal{F})(U) \longrightarrow \mathcal{S}(\mathcal{F})(V)$ is simply restriction of the discont. sec. $s|_V$.

There is an obvious embedding $\mathcal{F} \hookrightarrow \mathcal{S}(\mathcal{F})$.

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{S}(\mathcal{F})(U) \\ s & \longmapsto & \tilde{s} \end{array}$$

- $\tilde{s}(x) =$ germ at x determined by (U,s).
- $s \mapsto \tilde{s}$ is injective :- $\tilde{s} = 0 \Leftrightarrow \tilde{s}(x) = 0 \ \forall x \in U$

$\Leftrightarrow s$ is locally 0 on

$\mathcal{G}(\mathcal{F})$ - sometimes called the <u>Godement flabby envelope of $\mathcal{F}$</u>.

Observe that $\mathcal{G}(\mathcal{F})$ is indeed flabby :- $\mathcal{G}(\mathcal{F})(U) \twoheadrightarrow \mathcal{G}(\mathcal{F})(V)$.
If $\tilde{s} \in \mathcal{G}(\mathcal{F})(V)$, then just extend $\tilde{s}$ by 0 outside of V.

<u>Pf of ④</u> :- This follows from ③ by the "usual" argument :-

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, hook] & \mathcal{G}^0 \arrow[rr] \arrow[dr] & & \mathcal{G}^1 \arrow[rr] \arrow[dr] & & \mathcal{G}^2 \arrow[r] & \cdots \\
& & & \mathcal{G}^0/\mathrm{Im}(\mathcal{F}) \arrow[ur] & & \mathcal{G}^1/\mathrm{Im}(\mathcal{G}^0) \arrow[ur] & &
\end{tikzcd}$$

This is called <u>Godement's canonical flabby resolution of $\mathcal{F}$</u>.

The first Čech Cohomology group :-

This motivates the definition of Čech cohomology groups exactly like the calculation which motivated the definition of group cohomology.

The question is:- What is the obstruction to the exactness of the global sections functor.

Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of sheaves. Then upon taking global sections,

$$0 \to \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{H}(X) \text{ is exact.}$$

Let us see how this fails:-

In general, $\mathcal{G}(X) \longrightarrow \mathcal{H}(X)$ is not surjective.

Let $s \in \mathcal{H}(X)$.

Since $\mathcal{G} \twoheadrightarrow \mathcal{H}$, s can be locally lifted, i.e, $\exists$ open cover $\mathcal{U} = \{U_i\}$ of X $(X = \bigcup_i U_i)$, $\exists\, s_i \in \mathcal{G}(U_i)$ s.t. $s_i \longmapsto s|_{U_i}$ in $\mathcal{H}(U_i)$.

Consider, $\forall\, i, j$, $s_{ij} = s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in \mathcal{G}(U_i \cap U_j)$

Clearly, $s_{ij} \longmapsto 0$ in $\mathcal{H}(U_i \cap U_j)$

$\Rightarrow \exists\, t_{ij} \in \mathcal{F}(U_i \cap U_j)$ s.t. $t_{ij} \longmapsto s_{ij} = s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}$

(Since $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact $\forall\, U \underset{\text{open}}{\subseteq} X$)

So, ~~given an~~ for the open cover $\mathcal{U}$, we have $\{t_{ij}\}$ which satisfy the cocycle relation :-

$$t_{ij} + t_{jk} + t_{ki} = 0 \quad \text{on } U_i \cap U_j \cap U_k$$

or

$$\boxed{t_{ij} - t_{ik} + t_{jk} = 0 \quad \text{on } U_i \cap U_j \cap U_k}$$

Defn:- The 1-cocycles for the sheaf $\mathcal{F}$ ~~for~~ with respect to the covering $\mathcal{U}$ is defined as

$$Z^1(X, \mathcal{U}, \mathcal{F}) = \left\{ (t_{ij})_{ij \in I \times I} \;\middle|\; \begin{array}{l} \cdot\ t_{ij} \in F(U_i \cap U_j) \\ \cdot\ t_{ij} - t_{ik} + t_{jk} = 0 \text{ on } U_i \cap U_j \cap U_k \end{array} \right\}.$$

Defn:- The 1-coboundaries are:-

$$B^1(X, \mathcal{U}, \mathcal{F}) = \left\{ (t_{ij}) \in Z^1(X, \mathcal{U}, \mathcal{F}) \;\middle|\; \begin{array}{l} \exists\, s_i \in F(U_i) \text{ for which} \\ t_{ij} = s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \end{array} \right\}$$

Defn:- $\cdot$ $H^1(X, \mathcal{U}, \mathcal{F}) = Z^1(X, \mathcal{U}, \mathcal{F}) / B^1(X, \mathcal{U}, \mathcal{F})$

$\cdot$ $\check{H}^1(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^1(X, \mathcal{U}, \mathcal{F})$ Direct limit over all coverings $\mathcal{U}$, $\mathcal{U}_1 \leq \mathcal{U}_2$ if $\mathcal{U}_2$ is a refinement of $\mathcal{U}_1$.

$\check{H}^1(X, \mathcal{F})$ = first Čech cohomology group of X with coeffs. in $\mathcal{F}$.

We are looking at an "exact" sequence :-

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to \check{H}^1(X, \mathcal{F}) \to \check{H}^1(X, \mathcal{G}) \to \check{H}^1(X, \mathcal{F})$$

(Later we will define

- Čech cohomology groups $\check{H}^q(X, \mathcal{U}, \mathcal{F})$
- If X is reasonably nice, and $\mathcal{U}$ is fine enough then. $\check{H}^q(X, \mathcal{U}, \mathcal{F}) = H^q(X,