Lecture 18 (March-24)

Review of functoriality and some topological preliminaries for the base change theorem.

§1 Review. Let $f: X \longrightarrow Y$ be a continuous map.

(i) $f_*: S_X \longrightarrow S_Y$ is a left exact functor which maps injectives to injectives.

(A special case:- $A \overset{i}{\hookrightarrow} X$ closed embedding, i_* is an exact functor.)

(ii) $f^*: S_Y \longrightarrow S_X$ is an exact functor.

[Recall two basic, and possibly confusing, facts:-

- Definition of f^*:- $f^*\mathcal{G}$ is the sheafification of the presheaf $U \longmapsto {}^{p}f^*\mathcal{G}(U)$

$$ {}^{p}f^*\mathcal{G}(U) = \varinjlim_{f(U) \subset V} \mathcal{G}(V), \quad ({}^{p}f^*\mathcal{G})_x = \mathcal{G}_{f(x)}. $$

$$ \Rightarrow (f^*\mathcal{G})(U) = \left\{ \tilde{s}: U \longrightarrow \prod_{x \in U} \mathcal{G}_{f(x)} \;\middle|\; \begin{array}{l} \cdot\ \tilde{s}(x) \in \mathcal{G}_{f(x)} \\ \cdot\ \forall x \in U,\ \exists V_x\text{-open},\ x \in V_x \subset U \\ \exists W_x \underset{\text{open}}{\subset} Y,\ f(V_x) \subset W_x \\ \exists t \in \mathcal{G}(W_x) \text{ s.t.} \\ \tilde{s}(z) = t_{f(z)}\ \forall z \in V_x \end{array} \right\}. $$

- Adjunction formula

$$ \mathrm{Hom}_{S_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{S_Y}(\mathcal{G}, f_*\mathcal{F}). $$].

(iii) Functoriality-I.

Any continuous map $f: X \longrightarrow Y$ induces a homomorphism

$$ \boxed{H^q(Y, \mathcal{G}) \longrightarrow H^q(X, f^*\mathcal{G})} $$

for any sheaf $\mathcal{G}$ on Y.

Note:- For $q=0$, This means that we have a homomorphism

$$\mathcal{G}(Y) \longrightarrow f^*\mathcal{G}(X).$$

If $s \in \mathcal{G}(Y)$ then define $\tilde{s} : X \longrightarrow \prod_{x \in X} \mathcal{G}_{f(x)}$ as $\tilde{s}(x) = s_{f(x)}$, (

where $s_{f(x)}$ = germ at $f(x)$ determined by (s, Y).

(iv) Functoriality - II

$f : X \longrightarrow Y$ cont.

A - abelian group

$$\boxed{H^q(Y, A) \longrightarrow H^q(X, A).}$$

In other words, $X \longmapsto H^q(X, A)$ is a contravariant functor from TOP $\longrightarrow$ AB.

(v) Functoriality - III.

$f : X \longrightarrow Y$ cont.

$\mathcal{F}$ - sheaf on X

$$\boxed{H^q(Y, f_*\mathcal{F}) \longrightarrow H^q(X, \mathcal{F}).}$$

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This is because, functoriality says we have $H^q(Y, f_*\mathcal{F}) \longrightarrow H^q(X, f^*f_*\mathcal{F})$

Adjunction formula $\Rightarrow$ $\mathrm{Hom}_{S_X}(f^*f_*\mathcal{F}, \mathcal{F}) = \mathrm{Hom}_{S_Y}(f_*\mathcal{F}, f_*\mathcal{F}) \ni 1.$

$\Rightarrow$ $\exists$ canonical map $f^*f_*\mathcal{F} \longrightarrow \mathcal{F}$

$\Rightarrow$ $\exists$ " " $H^q(X, f^*f_*\mathcal{F}) \longrightarrow H^q(X, \mathcal{F}).$

(vi) Derived functors of f_* :-

Since $f_* : S_X \longrightarrow S_Y$ is a left exact functor we can consider its right derived functors : $R^q f_*$

Defn:- $R^q f_*(\mathcal{F}) = H^q(f_*\mathcal{J}^\bullet)$

where $0 \to \mathcal{F} \to \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots \to \mathcal{J}^n \to \cdots$ (

is an injective resolution of $\mathcal{F}$.

§2

Theorem

Let $f : X \longrightarrow Y$ be a continuous map and $\mathcal{F}$ - sheaf on X.

Suppose $\quad R^q f_*(\mathcal{F}) = 0 \quad \forall\, q \geq 1$

Then $\quad H^q(Y, f_*\mathcal{F}) \xrightarrow{\cong} H^q(X, \mathcal{F})$.

Proof :- Let $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots \to \mathcal{I}^n \to \cdots$ be an injective resolution of $\mathcal{F}$.

Consider $\quad 0 \to f_*\mathcal{F} \to f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to \cdots$

$$R^q f_*(\mathcal{F}) = 0 \implies H^q(f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \cdots) = 0 \quad \forall q \geq$$

$$\implies f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \cdots \text{ is exact}$$

f_* - left exact

$\implies \quad 0 \to f_*\mathcal{F} \to f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to \cdots$ is an injective resolution of $f_*\mathcal{F}$.

To get $H^q(Y, f_*\mathcal{F})$, apply $\Gamma(Y, -)$ and take cohomology :-

So consider

$$\begin{array}{ccccccc}
f_*\mathcal{I}^0(Y) & \longrightarrow & f_*\mathcal{I}^1(Y) & \longrightarrow & \cdots \longrightarrow & f_*\mathcal{I}^n(Y) \longrightarrow \cdots \\
\| & & \| & & & \\
\mathcal{I}^0(f^{-1}(Y)) & \longrightarrow & \mathcal{I}^1(f^{-1}(Y)) & \longrightarrow & \cdots & \\
\| & & \| & & & \\
\mathcal{I}^0(X) & \longrightarrow & \mathcal{I}^1(X) & \longrightarrow & \cdots &
\end{array}$$

$$H^q(Y, f_*\mathcal{F}) = H^q(f_*\mathcal{I}^\bullet(Y)) = H^q(\mathcal{I}^\bullet(X)) = H^q(X, \mathcal{F}).$$

Corollary :-

> Let $A \overset{i}{\hookrightarrow} X$ be a closed embedding.
>
> Let $\mathcal{F}$ be a sheaf on A. Then
>
> $$H^q(X, i_*\mathcal{F}) = H^q(A, \mathcal{F}).$$

Pf :- $A \overset{i}{\hookrightarrow} X$ closed embedding $\Rightarrow$ i_* - is exact

$\Rightarrow$ $R^q i_*(\mathcal{F}) = 0 \quad \forall\, q \geq 1$.

§3. **Basic Problem** :-

- How to determine if $R^q f_*(\mathcal{F}) = 0$?

or, more generally,

- How to compute $R^q f_*(\mathcal{F})$?

Answer (Base change theorem) :-

If X & Y are reasonably nice (This will be made precise) and $f: X \longrightarrow Y$ is a "proper map". Then we can compute the stalks of the derived functors $R^q f_*(\mathcal{F})$ via :-

$$\boxed{R^q f_*(\mathcal{F})_y \cong H^q(f^{-1}(y), i_y^*\mathcal{F}).}$$

when $i_y : f^{-1}(y) \hookrightarrow X$

§4 Some basic point-set topology :-

- **Defn**:- A top. space X is <u>locally compact</u> if every point has a relatively compact neighbourhood. , i.e.,

  $\forall x \in X$, $\exists U \underset{\text{open}}{\subset} X$ s.t. $x \in U$ and $\bar{U}$ is compact.

(**Ex**:- If X is locally compact and Hausdorff, then show that the set of relatively compact neighbourhoods at $x \in X$ forms a basis at x.)

- **Defn**:- Let $f : X \longrightarrow Y$ be a continuous map between top. spaces. We say f is a <u>proper map</u> if it pulls back compact sets to compact sets. , i.e., if $K \subset Y$ is compact then $f^{-1}(K)$ is compact.

  <u>Example</u> :- • Let $I = [0,1]$, $X \times [0,1] \xrightarrow{p_1} X$ is a proper map.

  • $\mathbb{R} \longrightarrow S^1$ $t \mapsto e^{2\pi i t}$ is not proper.

(**Ex**:- Let $f : X \longrightarrow Y$ be a ~~proper~~ continuous map of locally compact Hausdorff spaces.
Let X^+, Y^+ be the one-point compactifications of X and Y.
Show that f is proper $\Longleftrightarrow$ f extends to a continuous map $f^+ : X^+ \longrightarrow Y^+$.)

- **Defn**:- Let $X = \bigcup_{i \in I} U_i$ be an open covering.
  We say that it is a <u>locally finite</u> covering if $\forall x \in X$, $\exists$ open set V_x, $x \in V_x$, such that $V_x \cap U_i \neq \emptyset$ for only finitely many i

Defn:- (i) A <u>refinement</u> of an open cover $X = \bigcup_{i \in I} U_i$ is another open cover $X = \bigcup_{j \in J} V_j$ such that each V_j is in some U_i, i.e., $\exists$ function $\tau : J \longrightarrow I$ such that $V_j \subset U_{\tau(j)}$ $\forall j \in J$.

(ii) The refinement $X = \bigcup_{j \in J} V_j$ of $X = \bigcup_{i \in I} U_i$ is said to be a <u>strong refinement</u> if $\forall j$ $\overline{V_j} \subset U_{\tau(j)}$.

Defn:- X is <u>paracompact</u> if every open cover has a locally finite refinement.

(<u>Note</u>:- In Harder's book, paracompactness is defined as :- every open cover of <u>any open set</u> has a locally finite refinement.)

Ex:- If X is exhaustible by compact subsets, i.e., if $\exists$ sequence of compact subsets $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$ such that $X = \bigcup K_n$ and for all n, $K_n \subset \operatorname{int}(K_{n+1})$ then X is paracompact