Lecture-19
(March 26).

# Proper Base Change Theorem.

§1

Let $f : X \longrightarrow Y$ be a proper map of top. spaces.
Assume that $X$ is paracompact and locally compact, Hausdorff.
" " $Y$ is locally compact Hausdorff.
Let $\mathcal{F}$ be a sheaf on $X$. Then

$$R^q f_*(\mathcal{F})_y = H^q(f^{-1}(y), i_y^* \mathcal{F}). \quad \forall q \geq 0, \ \forall y \in Y$$

The notation is :-
- $f^{-1}(y)$ = fiber over $y$
- $i_y : f^{-1}(y) \hookrightarrow X$ inclusion.
- $i_y^* \mathcal{F}$ is the inverse image of $\mathcal{F}$ via $i_y^*$, it may also be denoted as $\mathcal{F}|_{f^{-1}(y)}$.
- $R^q f_*(\mathcal{F})_y$ = stalk at $y$ of the sheaf $R^q f_*(\mathcal{F})$.

§2 Two remarks before we start the proof :-

- The hypothesis on $X \xrightarrow{f} Y$ says that each fiber $f^{-1}(y)$ is compact and the embedding $f^{-1}(y) \hookrightarrow X$ has some particularly nice properties.

- The proof goes as:-
  - Check it for $q = 0$. $\left( f_*(\mathcal{F})_y = i_y^* \mathcal{F}(f^{-1}(y)). \right)$
  - Now prove it for all $q \geq 1$ by messing with an injective resolution for $\mathcal{F}$.

§3 ~~[illegible]~~

Corollary to proper base change theorem:-

Let $X \xrightarrow{f} Y$ be as in the base change theorem.

Let $\mathcal{F}$ be a sheaf on $X$.

Suppose none of the fibers of $f$ have nontrivial cohomology with respect to $\mathcal{F}$ then one of the maps induced by functoriality is an isomorphism, i.e.,

If $\quad H^q(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) = 0 \quad \forall q \geq 1 \quad \forall y \in Y$

Then $\quad H^q(Y, f_* \mathcal{F}) \xrightarrow{\sim} H^q(X, \mathcal{F})$.

Pf:- Use:- $\quad R^q f_*(\mathcal{F}) = 0 \quad \forall q \geq 1 \quad \Rightarrow \quad H^q(Y, f_*\mathcal{F}) \simeq H^q(X, \mathcal{F})$.

## §4 Nice Embeddings :-

Let $A \xhookrightarrow{i} X$ be a closed embedding.

(Meaning, $X$ is a top. space, $A$ is a closed subset, $i$ - inclusion map.)

Let $\mathcal{F}$ be a sheaf on $X$.

Recall :- $i^*\mathcal{F}$ = inverse image sheaf on $A$

= sheafification of the presheaf $V \longmapsto \varinjlim_{V \subset U} \mathcal{F}(U)$

$V \underset{\text{open}}{\subset} A$

$$i^*\mathcal{F}(V) = \left\{ s : V \longrightarrow \prod_{x \in V} \mathcal{F}_x \;\middle|\; \begin{array}{l} \cdot\ s(x) \in \mathcal{F}_x \quad \forall x \in V \\ \cdot\ \forall x \in V,\ \exists U_x \underset{\text{open}}{\subset} X,\ \exists t \in \mathcal{F}(U_x) \text{ s.t.} \\ \quad s(z) = t_z \quad \forall z \in U_x \cap V \end{array} \right\}.$$

~~Definition~~ In words, any section $s \in i^*\mathcal{F}(V)$ is locally the restriction of a section of $\mathcal{F}$.

**Definition**

A closed embedding $A \hookrightarrow X$ is said to be a <u>nice embedding</u> if

$\forall$ sheaves $\mathcal{F}$ on $X$

$\forall$ open subsets $V \subset A$

any section $s \in i^*\mathcal{F}(V)$ is in fact the restriction of a section of $\mathcal{F}$

i.e., $\forall s \in i^*\mathcal{F}(V)$, $\exists U \underset{\text{open}}{\subset} X$, $(V \subset U \cap A)$ $\exists t \in \mathcal{F}(U)$ s.t. $s(z) = t_z$ $\forall z \in V$

<u>Alternative definition</u> :-

$A \xhookrightarrow{i} X$ is a nice embedding if $\forall$ sheaves $\mathcal{F}$ on $X$ the presheaf inverse image $^{p}i^*\mathcal{F}$ is already a sheaf.

## Examples (Lemmas in Harder's book).

① If $X$ is a paracompact, locally compact Hausdorff space then any closed embedding $A \hookrightarrow X$ is nice.

② If $X$ is a Hausdorff top. space then $\forall\, x \in X$ the embedding $\{x\} \times I \hookrightarrow X \times I$ is nice.

(The second example is saying that the fibers of $X \times I \longrightarrow X$ are nicely embedded in $X \times I$. This will be used in the proof of the Homotopy axiom.)

### Consequences of a nice embedding

Let $A \overset{i}{\hookrightarrow} X$ be a nice embedding.

① Let $\mathcal{F}$ be any sheaf on $X$.

$$i^*\mathcal{F}(A) = \varinjlim_{A \subset U} \mathcal{F}(U).$$

(This is describing the global sections of the inverse image sheaf of $\mathcal{F}$)

② Let $\mathcal{I}$ be an injective sheaf on $X$ then $i^*\mathcal{I}$ is flabby and hence acyclic. In particular, if

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots \to \mathcal{I}^n \to \cdots$$

is an injective resolution of $\mathcal{F}$ in $S_X$ then its restriction to $A$:

$$0 \to i^*\mathcal{F} \to i^*\mathcal{I}^0 \to \cdots \to i^*\mathcal{I}^n \to \cdots$$

is an acyclic resolution of $i^*\mathcal{F}$ on $A$.

## §5 Proof of proper base change theorem:-

There is one more consequence of the hypothesis on $X \xrightarrow{f} Y$ :-

(★) For any $y \in Y$
For any neighbourhood $U$ of the fiber $f^{-1}(y) \subset U$
$\exists$ relatively compact neighbourhood $V$ of $y$ such that $f^{-1}(y) \subset f^{-1}(V) \subset U$

Pf of (★) :- Keep in mind that $Y$ is locally compact and $f$ is proper. and look at $(X-U) \cap f^{-1}(\bar{V})$ as $V$ runs through all rel. compact nbhds. of $y$.

- $(X-U) \cap f^{-1}(\bar{V})$ is compact ($f$ - proper, $\bar{V}$ - cpt.)
- $\bigcap_V (X-U) \cap f^{-1}(\bar{V})$ is empty ($x \in X-U$ check that $\exists V$ s.t $x \notin f^{-1}(\bar{V})$.)

$\Rightarrow$ $\exists$ one such $V$ for which $X-U \cap f^{-1}(\bar{V}) = \emptyset$ $\Rightarrow$ $f^{-1}(\bar{V}) \subset U$.

(Now we are ready to prove the theorem!)

Proof for $q=0$ :- $(R^0 f_* F)_y = f_* F_y$ , $H^0(f^{-1}(y), i_y^* F) = i_y^* F(f^{-1}(y))$

$$f_* F_y = \varinjlim_{y \in V} F(f^{-1}(V)) \quad \text{(by definition)}$$

$$= \varinjlim_{f^{-1}(y) \subset U} F(U) \quad \text{(by ★)}$$

$$= i_y^* F(f^{-1}(y)) \quad \text{(since } f^{-1}(y) \hookrightarrow X \text{ is a nice embedding)}$$

Proof for $q \geq 1$:-

Let $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots \to \mathcal{I}^n \to \cdots$

be an injective resolution of $\mathcal{F}$ in $S_X$.

Apply $f_*$ .. $0 \to f_*\mathcal{F} \to f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to \cdots$

This is a complex of injective sheaves. $\left( R^q f_*(\mathcal{F}) = H^q(f_*\mathcal{I}^\bullet) \right)$

$R^q f_*(\mathcal{F})_y = H^q(f_*\mathcal{I}^\bullet)_y = H^q(f_*\mathcal{I}^\bullet_y)$

Take the sequence of stalks at $y$:-

$$0 \to f_*\mathcal{F}_y \to f_*\mathcal{I}^0_y \to f_*\mathcal{I}^1_y \to \cdots$$

$R^q f_*(\mathcal{F})_y$ = cohomology of this complex.

But $f_*\mathcal{F}_y = i_y^*\mathcal{F}(f^{-1}(y))$ $(q=0)$

∴ This complex is the "same" as

$$0 \to i_y^*\mathcal{F}(f^{-1}(y)) \to i_y^*(\mathcal{I}^0(f^{-1}(y))) \to i_y^*\mathcal{I}^1(f^{-1}(y)) \to \cdots$$

which is the complex ~~obtained~~ of global sections for the complex

$$0 \to i_y^*\mathcal{F} \to i_y^*\mathcal{I}^0 \to i_y^*\mathcal{I}^1 \to \cdots$$

But $i_y : f^{-1}(y) \hookrightarrow X$ is a nice embedding $\Rightarrow$ this last complex is an acyclic resolution for $i_y^*\mathcal{F}$.

∴ $R^q f_*(\mathcal{F})_y = H^q(f_*\mathcal{I}^\bullet)_y = H^q(f_*\mathcal{I}^\bullet_y) = H^q(i_y^*\mathcal{I}^\bullet(f^{-1}(y))) = H^q(f^{-1}(y), \mathcal{F})$