# Spectral Sequences : Some basics.

Suppose we are given a resolution of a sheaf $\mathcal{F}$ :-

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots \to \mathcal{G}^p \to \cdots$$

and suppose this is not an acyclic resolution. The question is :-
To what extent can ~~the~~ we determine the cohomology of $\mathcal{F}$ from the cohomology groups of the sheaves $\mathcal{G}^\bullet$.

## §1 Injective resolution of ~~the~~ a resolution of $\mathcal{F}$ :-

To get started, we need the following lemma :-

**Lemma 1**

Let $0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots \to \mathcal{G}^p \to \cdots$ be a resolution of $\mathcal{F}$.

Then $\exists$ a diagram :-

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G}^0 \arrow[r] \arrow[d] & \mathcal{G}^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{G}^p \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & I^{00} \arrow[r] \arrow[d] & I^{10} \arrow[r] \arrow[d] & \cdots \arrow[r] & I^{p0} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & I^{01} \arrow[r] \arrow[d] & I^{11} \arrow[r] \arrow[d] & \cdots \arrow[r] & I^{p1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & & \vdots & \\
0 \arrow[r] & I^q \arrow[r] \arrow[d] & I^{0q} \arrow[r] \arrow[d] & I^{1q} \arrow[r] \arrow[d] & \cdots \arrow[r] & I^{pq} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & & \vdots &
\end{tikzcd}$$

$I^{pq}$ - $p^{th}$ column & $q^{th}$ row.

where

- Every vertical column is an injective resolution; $\begin{cases} I^\bullet \text{ is an inj. resol. of } \mathcal{F} \\ I^{p\bullet} \text{ '' '' '' '' } \mathcal{G}^p. \end{cases}$
- Every horizontal row is an exact sequence.

Proof :- The proof is the "same" as finding an injective resolution of a short exact sequence :- Take the resolution of $\mathcal{F}$

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots \to \mathcal{G}^p \to \cdots$$

and break it up as :-

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to K^0 \to 0$$

$$0 \to K^0 \to \mathcal{G}^1 \to K^1 \to 0$$

$$0 \to K^1 \to \mathcal{G}^2 \to K^2 \to 0 \quad \cdots \text{ etc..}$$

Find an injective resolution of each of these and splice them up. For example

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G}^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & I^0 \oplus J^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & I^1 \oplus J^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K^0 \arrow[r] \arrow[d] & \mathcal{G}^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^0 \arrow[r] \arrow[d] & J^0 \oplus R^0 \arrow[r] \arrow[d] & R^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^1 \arrow[r] \arrow[d] & J^1 \oplus R^1 \arrow[r] \arrow[d] & R^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

can be spliced along the column for $K^0$ to get :-

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G}^0 \arrow[r] \arrow[d] & \mathcal{G}^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & I^0 \oplus J^0 \arrow[r] \arrow[d] & J^0 \oplus R^0 \arrow[r] \arrow[d] & R^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & I^1 \oplus J^1 \arrow[r] \arrow[d] & J^1 \oplus R^1 \arrow[r] \arrow[d] & R^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

etc..

## §2 Double complexes & the simple complex:-

Now we hit the diagram in Lemma 1 by the global sections functor:-

The augmented double complex: - $\tilde{I}^{\bullet\bullet}(X)$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{G}^0(X) \arrow[r] \arrow[d] & \mathcal{G}^1(X) \arrow[r] \arrow[d] & \cdots \arrow[r] & & \mathcal{G}^p(X) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^0(X) \arrow[r] \arrow[d] & I^{00}(X) \arrow[r] \arrow[d] & I^{10}(X) \arrow[r] \arrow[d] & \cdots \arrow[r] & & I^{p0}(X) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^1(X) \arrow[r] \arrow[d] & I^{01}(X) \arrow[r] \arrow[d] & I^{11}(X) \arrow[r] \arrow[d] & \cdots \arrow[r] & & I^{p1}(X) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^2(X) \arrow[r] \arrow[d] & I^{02}(X) \arrow[r] \arrow[d] & I^{12}(X) \arrow[r] \arrow[d] & \cdots \arrow[r] & & I^{p2}(X) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & & & \vdots &
\end{tikzcd}$$

We also look at the non-augmented double complex $I^{\bullet\bullet}(X)$. :-

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & I^{00}(X) \arrow[r] \arrow[d] & I^{10}(X) \arrow[r] \arrow[d] & I^{20}(X) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^{p0}(X) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & I^{01}(X) \arrow[r] \arrow[d] & I^{11}(X) \arrow[r, "d'"] \arrow[d, "d''"] & I^{21}(X) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^{p1}(X) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & I^{02}(X) \arrow[r] \arrow[d] & I^{12}(X) \arrow[r] \arrow[d] & I^{22}(X) \arrow[r] \arrow[d] & \cdots \arrow[r] & I^{p2}(X) \arrow[r] \arrow[d] & {} \\
 & \vdots & \vdots & \vdots & & \vdots &
\end{tikzcd}$$

Horizontal differentials: $\quad d'^{p,q} : I^{p,q}(X) \longrightarrow I^{p+1,q}(X)$

vertical differentials. $\quad d''^{p,q} : I^{p,q}(X) \longrightarrow I^{p,q+1}(X)$

## The associated simple complex.

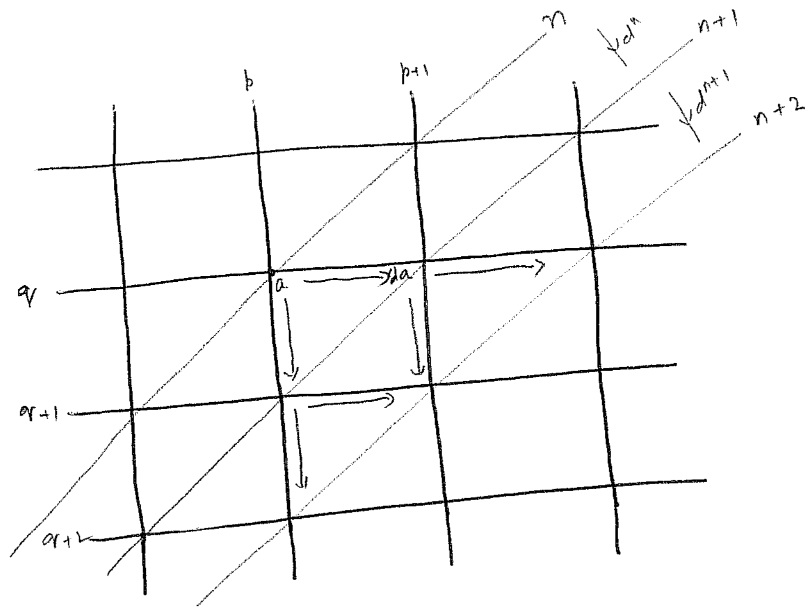
$$I^n_{simp}(X) := \bigoplus_{p+q=n} I^{p,q}(X) = \bigoplus_{p=0}^{n} I^{p,n-p}(X).$$

$$d^n : I^n_{simp}(X) \longrightarrow I^{n+1}_{simp}(X)$$

$$d^n = \sum_{p+q=n} {}'d^{p,q} + (-1)^p\, {}''d^{p,q}$$

**Lemma ②** $\boxed{(I^{\bullet}_{simp}(X),\ d^{\bullet}) \text{ is a complex. , i.e., } d^{n+1}\circ d^n = 0.}$

Pf:-




Take $a \in I^{p,q}(X)$ then chase it along the arrows shown:-

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[d, mapsto] & {}'d a \arrow[r, mapsto] \arrow[d, mapsto] & {}'d^2 a = 0 \\
(-1)^p d'' a \arrow[r, mapsto] \arrow[d, mapsto] & (-1)^{p+1} d'' d' a \quad (-1)^p d' d'' a & \\
(-1)^{2p} d'' d'' a = 0 & &
\end{tikzcd}$$

— These add up to zero, since $d''d' = d'd''$

## §3 Cohomology of various complexes within the double complex.

**Lemma ③**

① $H^q(X,\mathcal{F}) = H^q(I^\bullet(X))$.

② $H^q(X,\mathcal{G}^p) = H^q\big((I^{p\bullet}(X),\ {}''d^{p\bullet})\big)$.

③ $H^p(X,I^q) = H^p\big((I^{\bullet q}(X),\ {}'d^{\bullet q})\big) = \begin{cases} I^q(X), & p=0 \\ 0, & p\geq 1. \end{cases}$

All these assertions are by definition of sheaf cohomology!
For the last assertion, $I^q$ - injective $\Rightarrow$ acyclic!

**Lemma ④ (Cohomology of the simple complex).**

The inclusion $I^\bullet(X) \hookrightarrow I^\bullet_{simp}(X)$ induces an isomorphism :-

$$H^q(X,\mathcal{F}) \xrightarrow{\sim} H^q(I^\bullet_{simp}(X)).$$

Pf:- First of all, the inclusion $I^q(X) \hookrightarrow I^q_{simp}(X)$ is $I^q(X) \hookrightarrow I^{q,0}(X) \cap I^q_{simp}(X)$.

This inclusion is an inclusion of complexes:-

$$\begin{tikzcd} 0 \arrow[r] & I^q(X) \arrow[r] \arrow[d] & I^{q,0}(X) \arrow[r] \arrow[d] & {} \\ 0 \arrow[r] & I^{q+1}(X) \arrow[r] \arrow[d] & I^{q+1,0}(X) \arrow[d] & \\ & {} & {} & \end{tikzcd}$$

If $a \in I^q(X)$

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & {}'da \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\ {}''da \arrow[r, mapsto] & {}''d\,{}'da & \end{tikzcd}$$

To show that the induced map in cohomology is an isomorphism we check these two:-

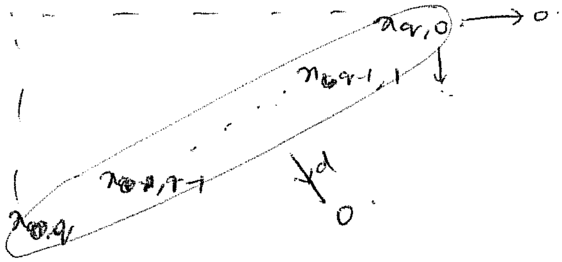
(i) $x \in Z^q(I^\bullet_{simp}(X))$ then we may modify $x$ by $B^q(I^\bullet_{simp}(X))$ so that $x$ looks like an element of $Z^q(I^\bullet(X))$

(ii) $x \in Z^q(I^\bullet(X))$ and if $x \in B^q(I^\bullet_{simp}(X))$ then $x \in B^q(I^\bullet(X))$.

For (i) :-

Let $x \in Z^q(I^\bullet_{simp}(X))$. Let us write $x = (x_{0,q}\,;\, x_{1,q-1}\,;\, \cdots\,;\, x_{q-1,1}\,;\, x_{q,0})$.

It looks more like :-



$d^q x = 0 \;\Rightarrow\; {}'d^{q,0}\, x_{q,0} = 0 \;\Rightarrow\; x_{q,0} \in Z^q(I^{\bullet 0}(X)) = B^q(I^{\bullet 0}(X))$

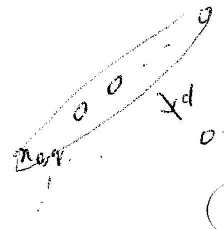
$\Rightarrow\; \exists\, y_{q-1,0} \in I^{q-1,0}(X) \ni x_{q,0} = {}'d^{q-1,0}\, y_{q-1,0}$

Replace $x$ by $x - d^q(0,0,\cdots,0,y_{q-1,0})$

and we may assume that $x$ looks like $x = (x_{0,q}\,;\cdots;\, x_{q-1,1}\,;\, 0)$.

Now work with $x_{q-1,1}$ ; and so on.

up to :: $x$ looks like $x = (x_{0,q}\,;\, 0,\cdots,0)$.



$d^q x = 0 \;\Rightarrow\; {}'d\, x_{0,q} = 0 \;\Rightarrow\; x_{0,q} \in I^q(X)$

$''d\, x_{0,q} = 0 \;\Rightarrow\; x_{0,q} \in Z^q(X)$.

(This says that $H^q(I^\bullet(X)) \longrightarrow H^q(I^\bullet_{simp}(X))$ )

For (ii), If $x_q \in Z^q(I^\bullet(X)) \cap B^q(I^\bullet_{simp}(X))$ then

$x = (\underset{=}{x_{0q}}, 0,\cdots,0)$, $''d\, x_{0q} = 0$,

Since $x \in B^q(I^\bullet_{simp}(X))$, $\exists\; y = (y_{0,q-1}\,;\,\cdots;\, y_{q-1,0}) \ni d^{q-1} y = x$

$\Rightarrow\; ''d\, y_{0,q-1} = x_{0,q}$

Now, $d^{q-1} y = x \;\Rightarrow\; {}'d\, y_{q-1,0} = 0 \;\Rightarrow\; \exists\, z_{q-2,0} \ni {}'d z_{q-2,0} = y_{q-1,0}$.

so replace $y$ by $y - d(0,\cdots,0,z_{q-2,0})$ & assume, $y = (y_{0,q-1},\cdots,y_{q-2,1},0)$

This does not change $d''\, y_{0,q-1} = x_{0,q}$.

continue if necessary, & assume $y = (y_{0,q-1}, 0,\cdots,0)$

Then $''d\, y_{0,q-1} = x_{0,q}$ & ${}'d\, y_{0,q-1} = 0$.

$\Rightarrow\; \exists\, x_{q-1} \in I^{q-1}(X)$ s.t. $y_{0,q-1}$ is image of $x_{q-1}$

$\Rightarrow\; ''d\, x_{q-1} = x$ i.e. $H^q(I^\bullet(X)) \hookrightarrow H^q(I^\bullet_{simp}(X))$

## §4 Edge homomorphism:

The inclusions $\mathcal{G}^{\bullet}(X) \hookrightarrow I^{\bullet\bullet}(X) \hookrightarrow I^{\bullet}_{simp}(X)$

gives an inclusion of complexes:- $\mathcal{G}^{\bullet}(X) \hookrightarrow I^{\bullet}_{simp}(X)$

which gives a homomorphism in cohomology

$$H^q(\mathcal{G}^{\bullet}(X)) \longrightarrow H^q(I^{\bullet}_{simp}(X)) \simeq H^q(X, \mathcal{F})$$

This gives the edge homomorphism:-

$$\boxed{H^q(\mathcal{G}^{\bullet}(X)) \longrightarrow H^q(X, \mathcal{F}).}$$

Ex:- Imitate the proof of Lemma ④ to prove:-

If $0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots$ is an acyclic resolution then the edge homomorphism is an isomorphism:-

$$H^q(\mathcal{G}^{\bullet}(X)) \xrightarrow{\simeq} H^q(X, \mathcal{F}).$$

Ex:- Show that there is an exact sequence:-

$$0 \longrightarrow H^1(\mathcal{G}^{\bullet}(X)) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \mathrm{Ker}(H^1(X, \mathcal{G}^0) \to H^1(X, \mathcal{G}^1)) \longrightarrow H^2(\mathcal{G}^{\bullet}(X))$$