Resolution of $\mathcal{F}$ $\rightsquigarrow$ Double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{G}^0(X) \arrow[r] \arrow[d] & \mathcal{G}^1(X) \arrow[r] \arrow[d] & \mathcal{G}^2 \arrow[r] & \cdots \\
0 \arrow[r] & \mathcal{C}^0(X) \arrow[r] \arrow[d] & \mathcal{C}^{00}(X) \arrow[r] \arrow[d] \arrow[dr, "d"] & \mathcal{C}^{10}(X) \arrow[r] \arrow[d] & & \\
0 \arrow[r] & \mathcal{C}^1(X) \arrow[r] \arrow[d] & \mathcal{C}^{01}(X) \arrow[r] \arrow[d] & \mathcal{C}^{11}(X) \arrow[r] \arrow[d] & & \\
 & \, & \, & \, & &
\end{tikzcd}$$

$$\mathcal{C}^n_{simp}(X) = \sum_{p+q=n} \mathcal{C}^{pq}(X)$$

$$d^n = \sum_{p+q=n} {}'d^{p,q} + (-1)^p\, {}''d^{p,q}$$

· The inclusion $\mathcal{C}^{\cdot}(X) \hookrightarrow \mathcal{C}^{\cdot}_{simp}(X)$ induces an isomorphism $H^q(X, \mathcal{F}) \simeq H^q(\mathcal{C}^{\cdot}_{simp}(X))$.

· We will look for filtrations:

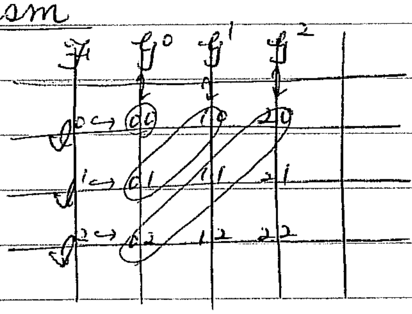
NOTATION: A: abelian group

By a filtration F^pA, one means a sequence

$$A = F^0A \supset F^1A \supset F^2A \supset \cdots$$

We will consider successive quotients: $F^pA / F^{p+1}A$: "pieces of A"

Edge homomorphism



$\mathcal{G}^{\cdot}(X) \hookrightarrow \mathcal{C}^{\cdot}_{simp}(X)$

$\rightsquigarrow H^n(\mathcal{G}^{\cdot}(X)) \longrightarrow H^n(\mathcal{C}^{\cdot}_{simp}(X)) \simeq H^n(X, \mathcal{F})$

: edge homomorphism

Lemma: If $\mathcal{G}^{\cdot}$ is an acyclic resolution, then the edge hom is an isom.

Consider any complex of sheaves:

$$0 \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots \to \mathcal{G}^n \to \cdots$$

<u>NOTE</u>: This complex has an inj. res., i.e. $\exists\ \mathcal{I}^{\cdot\cdot}$ s.t.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{G}^0 \arrow[r] \arrow[d] & \mathcal{G}^1 \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & \mathcal{I}^{00} \arrow[r] \arrow[d] & \mathcal{I}^{10} \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & \mathcal{I}^{01} \arrow[r] & \mathcal{I}^{11} \arrow[r] & {}
\end{tikzcd}$$

- Vertical columns are inj. resolutions.
- Horizontal ~~columns~~ rows are complexes.

~~Proof~~.

- $$\begin{tikzcd}
0 \arrow[r] & Z(\mathcal{G}^0) \arrow[r] \arrow[d] & \mathcal{G}^0 \arrow[r] & B(\mathcal{G}^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}^{\cdot} \arrow[r] & \mathcal{I}^{\cdot}\oplus\mathcal{J}^{\cdot} \arrow[r] & \mathcal{J}^{\cdot} \arrow[r] & 0
\end{tikzcd}$$

- $$\begin{tikzcd}
0 \arrow[r] & B(\mathcal{G}^1) \arrow[r] \arrow[d] & \boxed{Z(\mathcal{G}^1)} \arrow[r] \arrow[d] & H^1(\mathcal{G}^{\cdot}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}^{\cdot} \arrow[r] & \boxed{\mathcal{J}^{\cdot}\oplus\mathcal{K}^{\cdot}} \arrow[r] & \mathcal{K}^{\cdot} \arrow[r] & 0
\end{tikzcd}$$

- $$\begin{tikzcd}
0 \arrow[r] & \boxed{Z(\mathcal{G}^1)} \arrow[r] \arrow[d] & \mathcal{G}^1 \arrow[r] \arrow[d] & B(\mathcal{G}^2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \boxed{\mathcal{J}^{\cdot}\oplus\mathcal{K}^{\cdot}} \arrow[r] & \mathcal{J}^{\cdot}\oplus\mathcal{K}^{\cdot}\oplus\mathcal{L}^{\cdot} \arrow[r] & \mathcal{L}^{\cdot} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{G}^0 \arrow[r] \arrow[d] & \mathcal{G}^1 \arrow[r] \arrow[d] & \mathcal{G}^2 \\
0 \arrow[r] & \mathcal{I}^{\cdot}\oplus\mathcal{J}^{\cdot} \arrow[r] & \mathcal{J}^{\cdot}\oplus\mathcal{K}^{\cdot}\oplus\mathcal{L}^{\cdot} \arrow[r] & {}
\end{tikzcd}$$

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Now, consider the double complex $\mathcal{I}^{\cdot\cdot}(X)$ + the associated simple complex $\mathcal{I}_{simp}(X)$.

Our goal is to understand $H^n(\mathcal{I}_{simp}(X))$.

Want to define filtration of $H^n(\mathcal{I}_{simp}(X))$.

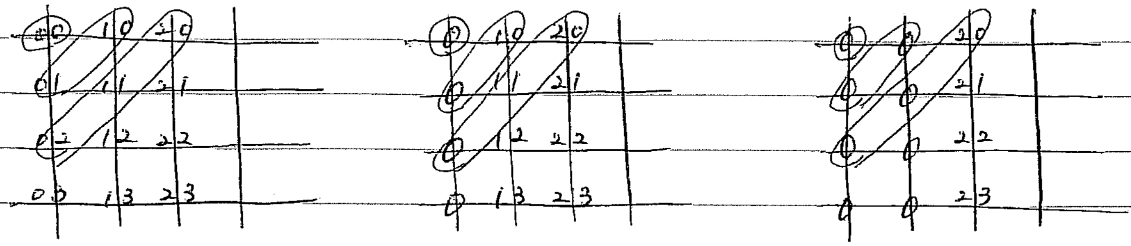
- Vertical filtrations.

Let $p \in \mathbb{Z}_{\geq 0}$

Consider the filtered double complex $F^p\mathcal{I}^{\cdot\cdot}(X)$, whose first $p-1$ columns are zero + starting from the p^{th} column, it is the same as $\mathcal{I}^{\cdot\cdot}(X)$. $\longrightarrow$

$$\cdot\ F^p\mathcal{A}^{\alpha\beta}(X) = \begin{cases} 0 & \alpha \le p-1 \\ \mathcal{A}^{\alpha\beta}(X) & \alpha \ge p \end{cases}$$

$\mathcal{A}^{\cdot\cdot}(X)$ $\qquad$ $F^1\mathcal{A}^{\cdot\cdot}(X)$ $\qquad$ $F^2\mathcal{A}^{\cdot\cdot}(X)$



Now, consider the associated simple complex $F^p\mathcal{A}^{\cdot}_{simp}(X)$

$$\begin{array}{l} \mathcal{A}^{\cdot}_{simp}(X):\ 0 \to \mathcal{A}^{00}(X) \to \mathcal{A}^{01}(X) \oplus \mathcal{A}^{10}(X) \to \mathcal{A}^{02}(X) \oplus \mathcal{A}^{11}(X) \oplus \mathcal{A}^{20}(X) \to \\ \quad \uparrow \\ F^1\mathcal{A}^{\cdot}_{simp}(X):\ 0 \to 0 \to \mathcal{A}^{10}(X) \to \mathcal{A}^{11}(X) \oplus \mathcal{A}^{20}(X) \to \cdots \\ \quad \uparrow \\ F^2\mathcal{A}^{\cdot}_{simp}(X):\ 0 \to 0 \to 0 \to \mathcal{A}^{20}(X) \to \mathcal{A}^{21}(X) \oplus \mathcal{A}^{30}(X) \to \cdots \end{array}$$

· Consider $H^n(F^p\mathcal{A}^{\cdot}_{simp}(X))$

Since $F^p\mathcal{A}^{\cdot}_{simp}(X) \hookrightarrow \mathcal{A}^{\cdot}_{simp}(X)$, one has a map in cohom,

$H^n(F^p\mathcal{A}^{\cdot}_{simp}(X)) \longrightarrow H^n(\mathcal{A}^{\cdot}_{simp}(X))$

NOTATION: $\mathrm{Image}\big(H^n(F^p\mathcal{A}^{\cdot}_{simp}(X)) \to H^n(\mathcal{A}^{\cdot}_{simp}(X))\big) =: F^pH^n(\mathcal{A}^{\cdot}_{simp}(X))$

· Consider a short exact seq. of complexes:

$$0 \longrightarrow F^{p+1}\mathcal{A}^{\cdot}_{simp}(X) \longrightarrow F^p\mathcal{A}^{\cdot}_{simp}(X) \longrightarrow F^p\mathcal{A}^{\cdot}_{simp}(X)/F^{p+1}\mathcal{A}^{\cdot}_{simp}(X) \longrightarrow 0$$

$\exists$ a long exact seq. in cohomology

$$\cdots \longrightarrow H^n(F^{p+1}\mathcal{A}^{\cdot}_{simp}(X)) \longrightarrow H^n(F^p\mathcal{A}^{\cdot}_{simp}(X)) \longrightarrow \underbrace{H^n\big(F^p\mathcal{A}^{\cdot}_{simp}(X)/F^{p+1}\mathcal{A}^{\cdot}_{simp}(X)\big)}_{\substack{\parallel \\ H^{n-p}(X,\, \mathcal{G}^p)}} \longrightarrow \cdots$$

$\longrightarrow$

- $\mathcal{E}_1$-term of the double complex : $E_1^{p,q} := H^q(X, \mathcal{G}^p)$.
  $d_1^{p,q} : E_1^{p,q} \longrightarrow E_1^{p+1,q}$
- $\mathcal{E}_2$-term : $E_2^{p,q} = \mathrm{Ker}(d_1^{p,q}) / \mathrm{Im}(d_1^{p-1,q})$
  $d_2^{p,q} = E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$
- $\mathcal{E}_r$-term : cohomology of $\mathcal{E}_{r-1}$, $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$