ecture-3

Abelian categories / Homology / Long exact sequences.

§1. Let R be a ring. (We will assume $1 \in R$ and all R-modules are unital.)

The category R-MOD has the following fundamental properties:

① R-MOD has a zero object 0. (Zero object = Initial & Final object

I is initial if $\mathrm{Hom}_{\mathcal{C}}(I, A)$ is a singleton $\forall A$

F is final if $\mathrm{Hom}_{\mathcal{C}}(A, F)$ is " " $\forall A$)

② $\forall A, B$, $\mathrm{Hom}_R(A,B)$ is an abelian group.

(The identity element $0_{A,B} \in \mathrm{Hom}_R(A,B)$ is the unique element
$$\begin{tikzcd} A \arrow[rr, "0_{A,B}"] \arrow[dr] & & B \\ & 0 \arrow[ur] & \end{tikzcd}$$)

③ composition distributes over addition in these abelian groups

$f(g_1+g_2) = fg_1 + fg_2$ $\qquad (f_1+f_2)g = f_1 g + f_2 g$.

④ Products & coproducts exist for any finite set of modules.

⑤ Every homomorphism has a kernel and cokernel.

$M \xrightarrow{f} N$ $\qquad \mathrm{Ker}(f) = \{ m \in M : f(m) = 0 \}$.

$\mathrm{Coker}(f) = N / \mathrm{Im}(f)$.

• Formal properties of $\mathrm{Ker}(f)$:-

(i) $K = \mathrm{Ker}(f) \overset{k}{\hookrightarrow} M$. $\qquad K \overset{k}{\hookrightarrow} M \xrightarrow{f} N$

(ii) $f \circ k = 0$

(iii) $\forall\, g : G \to M$, if $f \cdot g = 0$ then $\exists g' : G \to K$ such that $g = k g'$

$$\begin{tikzcd} K \arrow[r, hook] & M \arrow[r, "f"] & N \\ & G \arrow[ul, dashed, "g'"] \arrow[u, "g"] \arrow[ur, "0"'] & \end{tikzcd}$$

• Formal properties of $\mathrm{coker}(f) =: C$

(i) $N \overset{c}{\twoheadrightarrow} C$, $\qquad$ (ii) $M \xrightarrow{f} N \overset{c}{\twoheadrightarrow} C$

$f \cdot c = 0$

(iii)
$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "0"'] & N \arrow[r, two heads, "c"] \arrow[d, "h"] & C \arrow[dl, dashed] \\ & H & \end{tikzcd}$$

⑥ Every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

$\left(A \overset{i}{\hookrightarrow} B\right.$, $\operatorname{coker}(i)$:- $B \xrightarrow{c} B/A$, $\operatorname{Ker}(c) = A$ ... etc.$\left.\right)$

⑦ Every homomorphism $M \xrightarrow{f} N$ can be factored as $f = m \circ e$ where m is a monomorphism and e is an epimorphism.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "e"'] & & N \\ & f(M) \arrow[ur, "m"'] & \end{tikzcd}$$

<u>Definitions</u> :-

(i) A category which satisfies ① — ④ is called an <u>additive category</u>.

(ii) A category " " ① — ⑦ is called an <u>abelian category</u>.

(iii) Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. We say F is an <u>additive functor</u> if $\forall A, B \in Ob(\mathcal{C})$,

$$\operatorname{Hom}_{\mathcal{C}}(A, B) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(FA, FB)$$
$$f \longmapsto Ff$$

is a homomorphism of abelian groups, i.e., $F(f_1 + f_2) = Ff_1 + Ff_2$.

<u>Examples</u> :-

① R-MOD is an abelian category.

Special cases :-
- AB $= \mathbb{Z}$-MOD
- VECT_k = k-MOD (k-field)

} are abelian categories.

② R-MOD $\xrightarrow{*}$ AB The Hom-functors are additive functors.

Fix $M \in$ R-MOD, $\operatorname{Hom}(M, -)$: R-MOD $\longrightarrow$ AB.

§2 Complexes & Homology.

We will work in the category of R-modules.

Definition

(i) A complex $(M_\bullet, d_\bullet)$ is a sequence of modules $\{M_i\}$ and a sequence of homomorphism $d_i : M_i \longrightarrow M_{i-1}$ such that $d_{i-1} \circ d_i = 0$, or simply $d^2 = 0$.

$$\cdots \longleftarrow M_{i-1} \xleftarrow{d_i} M_i \xleftarrow{d_{i+1}} M_{i+1} \longleftarrow \cdots$$

(ii) If $(M_\bullet, d_\bullet)$ and $(M'_\bullet, d'_\bullet)$ are complexes, a homomorphism $\alpha : M_\bullet \longrightarrow M'_\bullet$ is a sequence of homomorphisms $\alpha_i : M_i \longrightarrow M'_i$ such that the diagram

$$\begin{tikzcd}
\cdots & M_{i-1} \arrow[l] \arrow[d, "\alpha_{i-1}"] & M_i \arrow[l, "d_i"'] \arrow[d, "\alpha_i"] & M_{i+1} \arrow[l, "d_{i+1}"'] \arrow[d, "\alpha_{i+1}"] & \cdots \arrow[l] \\
 & M'_{i-1} \arrow[l] & M'_i \arrow[l, "d'_i"'] & M'_{i+1} \arrow[l, "d'_{i+1}"'] & \cdots \arrow[l]
\end{tikzcd}$$

commutes, i.e., $\alpha_{i-1} \circ d_i = d'_i \circ \alpha_i$ or $\alpha \circ d = d' \circ \alpha$.

(iii) R-COMP :- The category of complexes of R-modules

Examples of complexes:-

① $\cdots \xleftarrow{0} M \xleftarrow{0} M \xleftarrow{0} M \xleftarrow{0} \cdots$ is a complex.

② Any short exact sequence $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$

- A sequence of maps $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be exact if $\mathrm{Ker}(g) = \mathrm{Im}(f)$
- To say $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ is a short exact sequence means
  - i) α is injective
  - ii) β is surjective
  - iii) $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$.

(The above examples might seem "special" or "artificial". We will seeing many many more examples of complexes)

Definition (Homology).

Let $(M_\bullet, d_\bullet)$ be a complex. $\cdots \leftarrow M_{i-1} \xleftarrow{d_i} M_i \xleftarrow{d_{i+1}} M_{i+1} \leftarrow \cdots$

The homology groups of $M_\bullet$ are:-

$$\boxed{H_i(M_\bullet) = \frac{\operatorname{Ker}(d_i : M_i \to M_{i-1})}{\operatorname{Im}(d_{i+1} : M_{i+1} \to M_i)}}$$

Elements of $\operatorname{Ker}(d_i)$ are called i-cycles and often we denote $Z_i(M_\bullet) = \operatorname{Ker}(d_i)$

" " $\operatorname{Im}(d_{i+1})$ " " i-boundaries " $B_i(M_\bullet) = \operatorname{Im}(d_{i+1})$

$\therefore$ $H_i(M_\bullet) = Z_i(M_\bullet) / B_i(M_\bullet)$

Note:- Fix $i \in \mathbb{Z}$. The i^{th} homology is a functor (in fact an additive functor) from the abelian category R-COMP to the abelian category R-MOD.

$$H_i : \text{R-COMP} \longrightarrow \text{R-MOD}.$$

observe that if we had a homomorphism $\alpha : M_\bullet \to M'_\bullet$ of complexes then we get a homomorphism $H_i(\alpha) : H_i(M_\bullet) \to H_i(M'_\bullet)$ of R-modules (You should draw a diagram and chase an element in $Z_i(M_\bullet)$ and also see what happens to elements of $B_i(M_\bullet)$.).

Conventions with indexing and Cohomology.

Consider a complex $\cdots \leftarrow M_{i-1} \leftarrow M_i \leftarrow M_{i+1} \leftarrow \cdots$

Here the indexing is assumed to run over all $i \in \mathbb{Z}$.

(i) We say $(M_\circ, d_\circ)$ is a <u>chain complex</u> if $M_i = 0 \ \forall i < 0$.

So a chain complex looks like :-

chain complex is also called a <u>positive complex</u>.

$$0 \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

(ii) We say $(M_\circ, d_\circ)$ is a <u>negative complex</u> if $M_i = 0 \ \forall i > 0$.

So a negative complex looks like

$$\cdots \leftarrow M_{-2} \leftarrow M_{-1} \leftarrow M_0 \leftarrow 0$$

It is traditional to change the indexing :- Put $M^i = M_{-i}$ and in this way we get a <u>cochain complex</u>.

$$0 \longrightarrow M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} M^3 \longrightarrow \cdots$$

(iii) Let (M°, d°) be a cochain complex. Then $\forall i \geq 0$

$$Z^i(M^\circ) = \operatorname{Ker}(d^i : M^i \longrightarrow M^{i+1}) = \underline{i\text{-cycles}}$$

$$B^i(M^\circ) = \operatorname{Im}(d^{i-1} : M^{i-1} \longrightarrow M^i) = \underline{i\text{-coboundaries}}$$

$$H^i(M^\circ) = Z^i(M^\circ)/B^i(M^\circ) = \underline{i^{th} \text{ cohomology group}}$$

- Chain complex is also called (homology) complex
- Cochain complex " " (cohomology) complex.
- The maps d_i or d^i are called <u>differentials</u>.

<u>Note</u> :- A chain complex $(M_\circ, d_\circ)$ is exact $\iff$ $H_i(M_\circ) = 0 \quad \forall i \geq 0$

A cochain complex (M°, d°) is exact $\iff$ $H^i(M^\circ) = 0 \quad \forall i \geq 0$.

Theorem (Long Exact homology sequence).

A short exact sequence of complexes gives a long exact homology sequence.

Given:-
$$0 \longrightarrow M'_\bullet \xrightarrow{\alpha} M_\bullet \xrightarrow{\beta} M''_\bullet \longrightarrow 0$$

(Short exact means for each i, $0 \to M'_i \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} M''_i \to 0$ is short exact),

we can define, $\forall i \in \mathbb{Z}$, a module homomorphism
$$\Delta_i : H_i(M''_\bullet) \longrightarrow H_{i-1}(M'_\bullet)$$
such that

the infinite sequence
$$\cdots \longrightarrow H_i(M'_\bullet) \xrightarrow{\tilde{\alpha}_i} H_i(M_\bullet) \xrightarrow{\tilde{\beta}_i} H_i(M''_\bullet) \xrightarrow{\Delta_i} H_{i-1}(M'_\bullet) \xrightarrow{\tilde{\alpha}_{i-1}} \cdots$$

is a long exact sequence.

"Proof":-

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & M'_{i+1} \arrow[r] \arrow[d] & M_{i+1} \arrow[r] \arrow[d] & M''_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_i \arrow[r] \arrow[d] & M_i \arrow[r, "\beta_i"] \arrow[d, "d_i"] & M''_i \arrow[r] \arrow[d, "d''_i"] & 0 \\
0 \arrow[r] & M'_{i-1} \arrow[r, "\alpha_{i-1}"] \arrow[d, "d'_{i-1}"] & M_{i-1} \arrow[r, "\beta_{i-1}"] \arrow[d, "d_{i-1}"] & M''_{i-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Let $z''_i \in Z_i(M''_\bullet)$. So $d''_i z''_i = 0$. $\exists\, m_i \in M_i \ni \beta_i m_i = z''_i$.

Now $\beta_{i-1} d_i m_i = d''_i \beta_i m_i = d''_i z''_i = 0 \Rightarrow d_i m_i \in \mathrm{Ker}(\beta_{i-1}) = \mathrm{Im}(\alpha_{i-1})$.

$\Rightarrow \exists!\, m'_{i-1} \in M'_{i-1} \ni \alpha_{i-1}(m'_{i-1}) = d_i m_i$

Now, $\alpha_{i-2} d'_{i-1} m'_{i-1} = d_{i-1} \alpha_{i-1} m'_{i-1} = d_{i-1} d_i m_i = 0 \Rightarrow d'_{i-1} m'_{i-1} = 0 \Rightarrow m'_{i-1} \in Z_{i-1}(M'_\bullet)$

$$\boxed{\Delta_i([z''_i]) = [\alpha_{i-1}^{-1} \circ d_i \circ \beta_i^{-1}(z''_i)]}$$

does the job!

Rest of this proof is an exercise!

The homomorphism $\Delta_i : H_i(M_\bullet'') \longrightarrow H_{i-1}(M_\bullet')$ is called the <u>connecting homomorphism</u>.

An important property :-

<u>The connecting homomorphism of the long exact homology sequence is natural</u>

This means that if we have two short exact sequences of complexes and a map between, then we have a map between the associated long exact sequences :-

Given

$$\begin{tikzcd} 0 \arrow[r] & M_\bullet' \arrow[r, "\alpha"] \arrow[d, "f'"] & M_\bullet \arrow[r, "\beta"] \arrow[d, "f"] & M_\bullet'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N_\bullet' \arrow[r, "\gamma"] & N_\bullet \arrow[r, "\delta"] & N_\bullet'' \arrow[r] & 0 \end{tikzcd}$$

which is commutative & has exact rows

then, the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_i(M_\bullet') \arrow[r, "\tilde{\alpha}_i"] \arrow[d, "\tilde{f}_i'"] & H_i(M_\bullet) \arrow[r, "\tilde{\beta}_i"] \arrow[d, "\tilde{f}_i"] & H_i(M_\bullet'') \arrow[r, "\Delta_i"] \arrow[d, "\tilde{f}_i''"] & H_{i-1}(M_\bullet') \arrow[r] \arrow[d, "\tilde{f}_{i-1}'"] & \cdots \\ \arrow[r] & H_i(N_\bullet') \arrow[r, "\tilde{\gamma}_i"] & H_i(N_\bullet) \arrow[r, "\tilde{\delta}_i"] & H_i(N_\bullet'') \arrow[r, "\Delta_i'"] & H_{i-1}(N_\bullet') \arrow[r] & \cdots \end{tikzcd}$$

is commutative.