

§1. Let R be a ring. (we will assume $1 \in R$ and all R -modules are unital.)
 The category $R\text{-MOD}$ has the following fundamental properties:

① $R\text{-MOD}$ has a zero object 0 . (Zero object = Initial & Final object)
 I is initial if $\text{Hom}_R(I, A)$ is a singleton $\forall A$
 F is final if $\text{Hom}_R(A, F)$ is " " $\forall A$

② $\forall A, B$, $\text{Hom}_R(A, B)$ is an abelian group.
 (The identity element $0_{A,B} \in \text{Hom}_R(A, B)$ is the unique element $\left. \begin{array}{c} A \xrightarrow{0_{A,B}} B \\ \searrow \quad \nearrow \\ 0 \end{array} \right)$

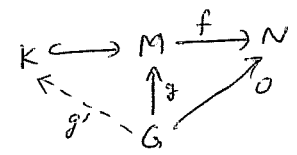
③ Composition distributes over addition in these abelian groups
 $f(g_1 + g_2) = fg_1 + fg_2$ $(f_1 + f_2)g = f_1g + f_2g$

④ Products & coproducts exist for any finite set of modules.

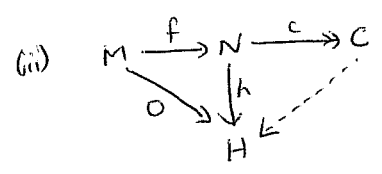
⑤ Every homomorphism has a kernel and cokernel.

$M \xrightarrow{f} N$ $\text{Ker}(f) = \{m \in M : f(m) = 0\}$
 $\text{coker}(f) = N / \text{Im}(f)$

Formal properties of $\text{Ker}(f)$:-
 (i) $K = \text{Ker}(f) \xrightarrow{c} M$
 $K \xrightarrow{c} M \xrightarrow{f} N$
 (ii) $f \circ c = 0$
 (iii) $\forall g: G \rightarrow M$, if $f \circ g = 0$ then
 $\exists g': G \rightarrow K$ such that $g = c \circ g'$



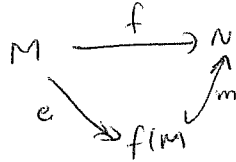
Formal properties of $\text{coker}(f) =: C$
 (i) $N \xrightarrow{c} C$, (ii) $M \xrightarrow{f} N \xrightarrow{c} C$
 $f \circ c = 0$



⑥ Every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

$$(A \xrightarrow{i} B, \text{coker}(i) := B \xrightarrow{c} B/A, \text{Ker}(c) = A \dots \text{etc.})$$

⑦ Every homomorphism $M \xrightarrow{f} N$ can be factored as $f = m \circ e$ where m is a monomorphism and e is an epimorphism.



Definitions:-

(i) A category which satisfies ①-④ is called an additive category.

(ii) A category " " ①-⑦ is called an abelian category.

(iii) Let \mathcal{C} and \mathcal{D} be additive categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ a functor.

we say F is an additive functor if \mathbb{R}

$$\forall A, B \in \text{Ob}(\mathcal{C}), \quad \begin{array}{ccc} \text{Hom}_{\mathcal{C}}(A, B) & \longrightarrow & \text{Hom}_{\mathcal{D}}(FA, FB) \\ f_1 \longrightarrow & & Ff \end{array} \quad \text{is a}$$

homomorphism of abelian groups, i.e., $F(f_1 + f_2) = Ff_1 + Ff_2$.

Examples:-

① $R\text{-MOD}$ is an abelian category.

Special cases:-

- $AB = \mathbb{Z}\text{-MOD}$
- $\text{VECT}_K = K\text{-MOD}$

(K -field) } are abelian categories.

② $R\text{-MOD} \xrightarrow{q} AB$ \forall Hom-functors are additive functors.

$$\forall M \in R\text{-MOD}, \quad \text{Hom}(M, -) : R\text{-MOD} \longrightarrow AB.$$

§2 Complexes & Homology.

we will work in the category of R -modules.

Definition

(i) A complex (M_\bullet, d_\bullet) is a sequence of modules $\{M_i\}$ and a sequence of homomorphisms $d_i: M_i \rightarrow M_{i-1}$ such that $d_{i-1} \circ d_i = 0$, or simply $d^2 = 0$.

$$\dots \longleftarrow M_{i-1} \xleftarrow{d_i} M_i \xleftarrow{d_{i+1}} M_{i+1} \longleftarrow \dots$$

(ii) If (M_\bullet, d_\bullet) and (M'_\bullet, d'_\bullet) are complexes, a homomorphism $\alpha: M_\bullet \rightarrow M'_\bullet$ is a sequence of homomorphisms $\alpha_i: M_i \rightarrow M'_i$ such that the diagram

$$\begin{array}{ccccccc} \dots & \longleftarrow & M_{i-1} & \xleftarrow{d_i} & M_i & \xleftarrow{d_{i+1}} & M_{i+1} & \longleftarrow & \dots \\ & & \downarrow \alpha_{i-1} & & \downarrow \alpha_i & & \downarrow \alpha_{i+1} & & \\ & & M'_{i-1} & \xleftarrow{d'_i} & M'_i & \xleftarrow{d'_{i+1}} & M'_{i+1} & \longleftarrow & \dots \end{array}$$

commutes, i.e., $d_{i-1} \circ d_i = d'_i \circ \alpha_i$ or $\alpha \circ d = d' \circ \alpha$.

(iii) R -COMP := The category of complexes of R -modules

Examples of complexes:-

① $\dots \xleftarrow{0} M \xleftarrow{0} M \xleftarrow{0} M \xleftarrow{0} \dots$

② Any short exact sequence $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$ is a complex.

A sequence of maps $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be exact if $\text{Ker}(g) = \text{Im}(f)$.

To say $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$ is a short exact sequence means

- (i) α is injective
- (ii) β is surjective
- (iii) $\text{Im}(\alpha) = \text{Ker}(\beta)$.

(The above examples might seem "special" or "artificial". We will see many more examples of complexes)

Definition (Homology).

Let (M_\bullet, d_\bullet) be a complex.

$$\cdots \leftarrow M_{i-1} \xleftarrow{d_i} M_i \xleftarrow{d_{i+1}} M_{i+1} \leftarrow \cdots$$

The homology groups of M_\bullet are:-

$$H_i(M_\bullet) = \frac{\text{Ker}(d_i : M_i \rightarrow M_{i-1})}{\text{Im}(d_{i+1} : M_{i+1} \rightarrow M_i)}$$

Elements of $\text{Ker}(d_i)$ are called i -cycles and often we denote $Z_i(M_\bullet) = \text{Ker}(d_i)$
" " $\text{Im}(d_{i+1})$ " " i -boundaries " " $B_i(M_\bullet) = \text{Im}(d_{i+1})$

$$\therefore H_i(M_\bullet) = Z_i(M_\bullet) / B_i(M_\bullet)$$

Note:- For $i \in \mathbb{Z}$. The i^{th} homology is a functor (in fact an additive functor) from the abelian category $R\text{-COMP}$ to the abelian category $R\text{-MOD}$.

$$H_i : R\text{-COMP} \longrightarrow R\text{-MOD}.$$

observe that if we had a homomorphism $\alpha : M_\bullet \rightarrow M'_\bullet$ of complexes then we get a homomorphism $H_i(\alpha) : H_i(M_\bullet) \rightarrow H_i(M'_\bullet)$ of R -modules (you should draw a diagram and chase an element in $Z_i(M_\bullet)$ and also see what happens to elements of $B_i(M_\bullet)$).

Conventions with indexing and cohomology.

Consider a complex

$$\dots \leftarrow M_{i-1} \leftarrow M_i \leftarrow M_{i+1} \leftarrow \dots$$

Here the indexing is assumed to run over all $i \in \mathbb{Z}$.

(i) We say (M_\bullet, d_\bullet) is a chain complex if $M_i = 0 \quad \forall i < 0$.

So a chain complex looks like:-

$$0 \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

chain complex is also called a positive complex.

(ii) We say (M_\bullet, d_\bullet) is a negative complex if $M_i = 0 \quad \forall i > 0$.

So a negative complex looks like

$$\dots \leftarrow M_{-2} \leftarrow M_{-1} \leftarrow M_0 \leftarrow 0$$

It is traditional to change the indexing \therefore Put $M^i = M_{-i}$ and in this way we get a cochain complex.

$$0 \rightarrow M_0^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} M^3 \rightarrow \dots$$

(iii) Let (M^\bullet, d^\bullet) be a cochain complex. Then $\forall i \geq 0$

$$Z^i(M^\bullet) = \text{Ker}(d^i: M^i \rightarrow M^{i+1}) = \underline{i\text{-cycles}}$$

$$B^i(M^\bullet) = \text{Im}(d^{i-1}: M^{i-1} \rightarrow M^i) = \underline{i\text{-coboundaries}}$$

$$H^i(M^\bullet) = Z^i(M^\bullet) / B^i(M^\bullet) = \underline{i^{\text{th}} \text{ cohomology group.}}$$

- Chain complex is also called (homology) complex
- Cochain complex " " (cohomology) complex.
- The maps d_i or d^i are called differentials.

NOTE:- A chain complex (M_\bullet, d_\bullet) is exact $\Leftrightarrow H_i(M_\bullet) = 0 \quad \forall i \geq 0$

A cochain complex (M^\bullet, d^\bullet) is exact $\Leftrightarrow H^i(M^\bullet) = 0 \quad \forall i \geq 0$

Theorem (Long Exact homology sequence).

A short exact sequence of complexes gives a long exact homology sequence.

Given: $0 \rightarrow M'_0 \xrightarrow{\alpha} M_0 \xrightarrow{\beta} M''_0 \rightarrow 0$

(Short exact means for each i , $0 \rightarrow M'_i \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} M''_i \rightarrow 0$ is short exact),

we can define, $\forall i \in \mathbb{Z}$, a module homomorphism

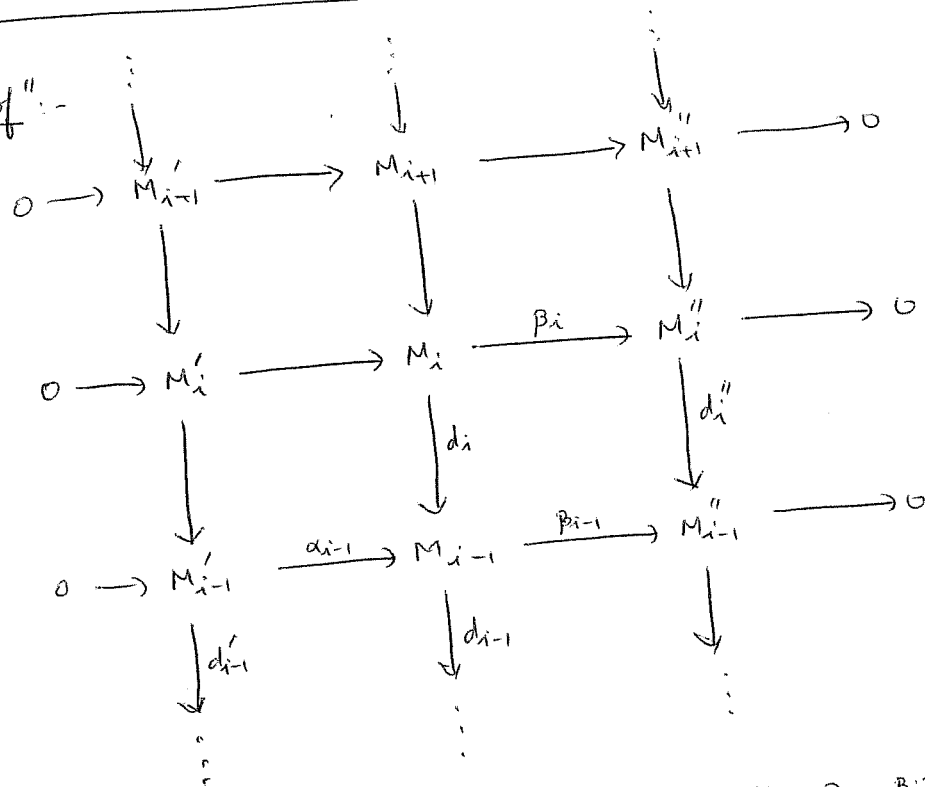
$$\Delta_i : H_i(M''_0) \longrightarrow H_{i-1}(M'_0) \text{ such that}$$

the infinite sequence

$$\dots \rightarrow H_i(M'_0) \xrightarrow{\tilde{\alpha}_i} H_i(M_0) \xrightarrow{\tilde{\beta}_i} H_i(M''_0) \xrightarrow{\Delta_i} H_{i-1}(M'_0) \xrightarrow{\tilde{\alpha}_{i-1}} \dots$$

is a long exact sequence.

"Proof":-



Let $z''_i \in Z_i(M''_0)$. So $d''_i z''_i = 0$. $\exists m_i \in M_i \ni \beta_i m_i = z''_i$.

Now $\beta_{i-1} d_i m_i = d''_i \beta_i m_i = d''_i z''_i = 0 \Rightarrow d_i m_i \in \text{Ker}(\beta_{i-1}) = \text{Im}(d_{i-1})$.

$\Rightarrow \exists! m'_{i-1} \in M'_{i-1} \ni \alpha_{i-1}(m'_{i-1}) = d_i m_i$

Now, $\alpha_{i-2} d'_{i-1} m'_{i-1} = d_{i-1} \alpha_{i-1} m'_{i-1} = d_{i-1} d_i m_i = 0 \Rightarrow d'_{i-1} m'_{i-1} = 0 \Rightarrow m'_{i-1} \in Z_{i-1}(M'_0)$

$$\Delta_i([z''_i]) = [\alpha_{i-1}^{-1} \circ d_i \circ \beta_i^{-1}(z''_i)]$$

does the job!

Rest of this proof is an exercise!

The homomorphism $\Delta_i: H_i(M_0'') \rightarrow H_{i-1}(M_0')$ is called the connecting homomorphism.

An important property:-

The connecting homomorphism of the long exact homology sequence is natural

This means that if we have two short exact sequences of complexes and a map between them, then we have a map between the associated long exact sequences:-

Given

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M_0' & \xrightarrow{\alpha} & M_0 & \xrightarrow{\beta} & M_0'' \longrightarrow 0 \\
 & & \downarrow f' & & \downarrow f & & \downarrow f'' \\
 0 & \longrightarrow & N_0' & \xrightarrow{\gamma} & N_0 & \xrightarrow{\delta} & N_0'' \longrightarrow 0
 \end{array}$$

which is commutative & has exact rows

then, the diagram

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & H_i(M_0') & \xrightarrow{\tilde{\alpha}_i} & H_i(M_0) & \xrightarrow{\tilde{\beta}_i} & H_i(M_0'') & \xrightarrow{\Delta_i} & H_{i-1}(M_0') & \longrightarrow & \dots \\
 & & \downarrow \tilde{f}_i & & \downarrow \tilde{f}_i & & \downarrow \tilde{f}_i'' & & \downarrow \tilde{f}_{i-1}' & & \\
 \dots & \longrightarrow & H_i(N_0') & \xrightarrow{\tilde{\gamma}_i} & H_i(N_0) & \xrightarrow{\tilde{\delta}_i} & H_i(N_0'') & \xrightarrow{\Delta_i} & H_{i-1}(N_0') & \longrightarrow & \dots
 \end{array}$$

is commutative.

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