Lecture 5 & 6

DERIVED FUNCTORS.

§1 Let $F : R\text{-MOD} \longrightarrow AB$ be an additive functor.

Here is a very quick summary of what we will do :-

F - covariant

① Given a module M, work with a projective resolution of M, and define the left derived functors $L_nF : R\text{-MOD} \longrightarrow AB$.

A short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of modules gives a long exact sequence

$$\cdots \longrightarrow L_2FC \longrightarrow L_1FA \longrightarrow L_1FB \longrightarrow L_1FC \longrightarrow L_0FA \longrightarrow L_0FB \longrightarrow L_0FC \longrightarrow 0$$

② Similarly, working with injective resolutions. we get the right derived functors. $R^nF : R\text{-MOD} \longrightarrow AB$. Given a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we get a long exact sequence

$$0 \longrightarrow R^0FA \longrightarrow R^0FB \longrightarrow R^0FC \longrightarrow R^1FA \longrightarrow R^1FB \longrightarrow R^1FC \longrightarrow R^2FA \longrightarrow \cdots$$

G- contravariant

① Use projective resolutions --- to get --- right derived functors.

Now a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives

$$0 \longrightarrow R^0GC \longrightarrow R^0GB \longrightarrow R^0GA \longrightarrow R^1GC \longrightarrow \cdots$$

② Injective resolutions ---- left derived functors... etc.

clearly, for the construction of derived functors we need some information about projective & injective resolutions.

§2 Theorem (Basic facts on Projective resolutions).

① Every module has a projective resolution.

Given M, $\exists$ $(P_\bullet, \varepsilon)$, each P_i is free, $0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_n \leftarrow$

② A homomorphism of modules can be lifted to a homomorphism of their projective resolutions; two such lifts are homotopic.

Given $M \xrightarrow{f} N$, given resolution $M \leftarrow P_\bullet$, $N \leftarrow Q_\bullet$; $\exists\, \alpha : P_\bullet \to Q_\bullet$ such that

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l] \arrow[d, "\alpha_0"] & P_1 \arrow[l] \arrow[d, "\alpha_1"] & \cdots \arrow[l] \\
0 & N \arrow[l] & Q_0 \arrow[l] & Q_1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

③ Any short exact sequence of modules has a short exact sequence of projective resolutions.

Given $0 \to A \to B \to C \to 0$ of modules,

$\exists$ projective res. $A \xleftarrow{\varepsilon} P_\bullet$, $B \xleftarrow{\theta} Q_\bullet$, $C \xleftarrow{\eta} R_\bullet$ and a short exact sequence $0 \to P_\bullet \xrightarrow{\imath} Q_\bullet \xrightarrow{\pi} R_\bullet \to 0$ of complexes.

So we are looking at diagram:-

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_2 \arrow[r] \arrow[d] & Q_2 \arrow[r] \arrow[d] & R_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & R_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & Q_0 \arrow[r] \arrow[d, "\theta"] & R_0 \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

- where all rows are exact.
- all the vertical columns are the projective resolutions

Proof :- we have already proved ① & ②.

For ③ :- Start with $0 \to A \to B \to C \to 0$.

Take P_0, R_0 projective such that

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\varepsilon"'] & P_0 \oplus R_0 \arrow[r] \arrow[d, dashed] & R_0 \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P_0 \oplus R_0 \arrow[dl, dashed, "\eta^*"'] \arrow[d] \\
 & R_0 \arrow[d, "\eta"] \\
B \arrow[r, "g"] & C
\end{tikzcd}$$

The question is how to define $P_0 \oplus Q_0 \twoheadrightarrow B$?

use $P_0 \oplus R_0 \to R_0 \xrightarrow{\eta} C$ & projectivity of $P_0 \oplus R_0$ to get an $\eta^* : P_0 \oplus R_0 \to B$

Define $\theta : P_0 \oplus R_0 \longrightarrow B$ as $\theta(p_0, r_0) = \eta^*(p_0, r_0) + f\varepsilon p_0$.

Now check that we have

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\varepsilon"'] & Q_0 \arrow[r] \arrow[d, "\theta"] & R_0 \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with $Q_0 = P_0 \oplus R_0$

By the snake lemma (or otherwise), one has an exact sequence

$$0 \to \mathrm{Ker}(\varepsilon) \to \mathrm{Ker}(\theta) \to \mathrm{Ker}(\eta) \to 0.$$

Get hold of a "projective" sequence

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & R_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\varepsilon) \arrow[r] \arrow[d, hook] & \mathrm{Ker}(\theta) \arrow[r] \arrow[d] & \mathrm{Ker}(\eta) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] & Q_0 \arrow[r] & R_0 \arrow[r] & 0
\end{tikzcd}$$

Now forget the kernels to look at :

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & R_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & R_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

By induction, we are done!

§3 ## Recipe for left derived functors of a covariant functor.

Let $F : R\text{-MOD} \longrightarrow AB$ be a covariant functor.
additive

- Let M be an R-module.

Take a projective resolution of M

$$0 \leftarrow M \xleftarrow{\varepsilon} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} \cdots \leftarrow P_n \leftarrow \cdots$$

Apply F to get

$$0 \leftarrow FM \xleftarrow{F\varepsilon} FP_0 \xleftarrow{Fd_1} FP_1 \leftarrow \cdots$$

Consider the complex

$$FP_0 \xleftarrow{Fd_1} FP_1 \xleftarrow{Fd_2} FP_2 \leftarrow \cdots$$

Take homology of this complex to get

$$\boxed{L_nF(M) = H_n(FP_\cdot)} \qquad n \geq 0.$$

$$L_0F(M) = \frac{FP_0}{\mathrm{Im}(Fd_1)}$$

- Let $M \xrightarrow{f} N$ be a homomorphism of modules.

Take projective resolutions of M & N, and lift f to an $\alpha : P_\cdot \to Q_\cdot$

We are looking at :-

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "f"] & P_0 \arrow[l] \arrow[d, "\alpha_0"] & P_1 \arrow[l] \arrow[d, "\alpha_1"] & \arrow[l] \\
0 & N \arrow[l] & Q_0 \arrow[l] & Q_1 \arrow[l] & \arrow[l]
\end{tikzcd}$$

Apply F, and consider the map $F\alpha_\cdot : FP_\cdot \to FQ_\cdot$, which looks like

$$\begin{tikzcd}
FP_0 \arrow[d, "F\alpha_0"] & FP_1 \arrow[l] \arrow[d, "F\alpha_n"] & FP_2 \arrow[l] \arrow[d, "F\alpha_n"] & \cdots \arrow[l] \\
FQ_0 & FQ_1 \arrow[l] & FQ_2 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Take the map induced in ~~cohom~~ homology :- $\widetilde{F\alpha_n} : H_n(FP_\cdot) \to H_n(FQ_\cdot)$

$$\boxed{L_nF(f) = \widetilde{F\alpha_n}}$$

One has to check many details:-

(i) Well-definedness :- All the assertions about homotopy of maps gives well-definedness; like independence of choice projective resolution, independence of the lift α of f.

(ii) $\forall\, n \geq 0$ $\quad L_n F$ is an additive covariant functor.

- To get the long exact sequence (which justifies the terminology <u>left derived fun</u>

Start with a short exact sequence $0 \to A \to B \to C \to 0$ of modules.

Take a projective resolution of this sequence

$$
\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & R_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
$$

Apply F, and consider

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & FP_2 \arrow[r] \arrow[d] & FQ_2 \arrow[r] \arrow[d] & FR_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & FP_1 \arrow[r] \arrow[d] & FQ_1 \arrow[r] \arrow[d] & FR_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & FP_0 \arrow[r] & FQ_0 \arrow[r] & FR_0 \arrow[r] & 0
\end{tikzcd}
$$

Take the long exact homology sequence associated to this short exact sequence of complexes. (Note:- R_n - projective $\Rightarrow$ $Q_n \simeq P_n \oplus R_n$ $\Rightarrow$ $FQ_n = FP_n \oplus FR_n$ $\Rightarrow$ each row is exact.) we will get the long exact sequence:-

$$\cdots \longrightarrow L_2FC \longrightarrow L_1FA \longrightarrow L_1FB \longrightarrow L_1FC \longrightarrow L_0FA \longrightarrow L_0FB \longrightarrow L_0FC \longrightarrow 0$$

Now for the right derived functors :- Simply "reverse" all arrows.
It is best to go through the recipe one more time. The categorical "dual" of projective is injective. Here are some basic facts about injective resolutions.

4 **Theorem (Injective resolutions).**

① Every module M has an injective resolution $I^\bullet$, i.e., $\exists$ exact sequence

$$0 \to M \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$$

with all the I^n being injective.

② A homomorphism $f: M \to N$ of modules can be lifted to a homomorphism of their injective resolutions :- Given $f: M \to N$, $0 \to M \to I^\bullet$, $0 \to N \to J^\bullet$, we have $\alpha: I^\bullet \to J^\bullet$ such that

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & I^0 \arrow[r] \arrow[d, "\alpha^0"] & I^1 \arrow[r] \arrow[d, "\alpha^1"] & I^2 \arrow[r] \arrow[d, "\alpha^2"] & \cdots \\
0 \arrow[r] & N \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & J^2 \arrow[r] & \cdots
\end{tikzcd}$$

Two such lifts α and β of f are homotopic. $\therefore$ ($\exists\, s^n : I^n \to J^{n-1}$ s.t. "$sd + ds = \alpha - \beta$".)

③ A short exact sequence of modules $0 \to A \to B \to C \to 0$ admits a short exact sequence of injective resolutions, i.e., $\exists$ injective resolution $0 \to A \to I^\bullet$, $0 \to B \to J^\bullet$, $0 \to C \to K^\bullet$ such that we have

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^2 \arrow[r] \arrow[d] & J^2 \arrow[r] \arrow[d] & K^2 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

- where all rows are exact
- all vertical columns are the injective resolutions.

§5 Recipe for right derived functors of a covariant functor F :-

- Given a module M.

Take an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

Apply F, and consider

$$FI^0 \to FI^1 \to FI^2 \to \cdots$$

(This is a cochain complex.)

Take cohomology of this complex to define :-

$$\boxed{R^nF(M) = H^n(FI^\bullet)} \quad n \geq 0 \quad ; \quad R^0F(M) = \mathrm{Ker}(FI^0 \to$$

(It is a good exercise to check that this is well-defined. Take another resolution $0 \to M \to J^0$, now lift $1_M : M \to M$ in both directions to get $\alpha^\bullet : I^\bullet \to J^\bullet$ & $\beta^\bullet : J^\bullet \to$ and mess around with induced maps in cohomology .... etc... Have fun!)

- Given a homomorphism $f : M \to N$ of modules.

Lift f to a homomorphisms $\alpha^\bullet : I^\bullet \to J^\bullet$ of resolutions of M & N.

So we are looking at

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & I^0 \arrow[r] \arrow[d, "\alpha^0"] & I^1 \arrow[r] \arrow[d, "\alpha^1"] & \cdots \\ 0 \arrow[r] & N \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots \end{tikzcd}$$

Apply F, and consider

$$\begin{tikzcd} FI^0 \arrow[r] \arrow[d, "F\alpha^0"'] & FI^1 \arrow[r] \arrow[d, "F\alpha^1"'] & FI^2 \arrow[r] \arrow[d, "F\alpha^n"'] & \cdots \\ FJ^0 \arrow[r] & FJ^0 \arrow[r] & FJ^2 \arrow[r] & \cdots \end{tikzcd}$$

$F\alpha^\bullet : FI^\bullet \to FJ^\bullet$

Now take the map induced in cohomology :-

$$\boxed{R^nF(f) : R^nF(M) \to R^nF(N) \quad \text{is} \quad R^nF(f) = \widetilde{F\alpha^n}}$$

- A short exact sequence of modules

$$0 \to A \to B \to C \to 0$$

yields a long exact sequence

$$0 \to R^0FA \to R^0FB \to R^0FC \to R^1FA \to R^1FB \to R^1FC \to \cdots$$

as follows:-

Take a short exact sequence of resolutions :-

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I^\bullet & \to & J^\bullet & \to & K^\bullet & \to & 0 \end{array}$$

Apply F and consider:-

$$\begin{array}{ccccccccc} 0 & \to & FI^0 & \to & FJ^0 & \to & FK^0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & FI^1 & \to & FJ^1 & \to & FK^1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & FI^2 & \to & FJ^2 & \to & FK^2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Consider $0 \to I^n \to J^n \to K^n$

Since I^n is injective, the sequence splits

$\Rightarrow J^n \simeq I^n \oplus K^n$

$\Rightarrow FJ^n \simeq FI^n \oplus FK^n$.

$\Rightarrow$ All rows are exact.

All columns are only cochain complexes.

Take the long exact cohomology sequence:-

$\exists$ $\Delta^n : \underset{R^nF(C)}{\underset{\parallel}{H^n(FK^\bullet)}} \to \underset{R^{n+1}F(A)}{\underset{\parallel}{H^{n+1}(FI^\bullet)}}$ such that the sequence

$$0 \to R^0F(A) \to R^0F(B) \to R^0F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \cdots$$

is exact.

§6 Modifications needed for a contravariant functor $G : R\text{-MOD} \to AB$.

§§. Left derived functors of G :-

To define $L_n G(M)$ start with an injective resolution of M :-

$$0 \to M \to I^0 \to I^1 \to \cdots$$

Apply G and consider

$$GI^0 \leftarrow GI^1 \leftarrow GI^2 \leftarrow \cdots$$

(Note that this is a chain comp

Take homology to get

$$\boxed{L_n G(M) = H_n(GI^\circ)}$$

Now everything else will fall in place. For example, given

$$0 \to A \to B \to C \to 0$$

we get a long exact sequence :-

$$0 \leftarrow L_0GA \leftarrow L_0GB \leftarrow L_0GC \leftarrow L_1GA \leftarrow L_1GB \leftarrow L_1GC \leftarrow L_2GA \leftarrow \cdots$$

§§ Right derived functors of G.

Take a projective resolution of M (& then follow your nose!)

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

Apply G, and consider

(this is a cochain complex)

$$GP_0 \to GP_1 \to GP_2 \to \cdots$$

Take cohomology of this complex to get :-

$$\boxed{R^n G(M) = H^n(GP_\circ).}$$

... etc ...