Lecture-7

The Ext-functors.

The EXT-functors are the most important functors for this course!
(And arguably in all of homological algebra.)

§ Definition:-

Fix a module N.

Consider the functor $G = \mathrm{Hom}(-,N) : R\text{-MOD} \longrightarrow AB$.

Note that G is a left exact contravariant functor.

(Here is an example to show that G is not exact:- Take $R = \mathbb{Z}$. Then $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ is not exact, because apply it to the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

to get

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) & \xrightarrow{n} & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) & , \\ & \| & & \cong & & \cong & \\ & 0 & & \mathbb{Z} & & \mathbb{Z} & \end{array}$$

and clearly it is not exact on the right!)

The right derived functors of $\mathrm{Hom}(-,N)$ are the Ext-functors, i.e.,

$$\boxed{\mathrm{Ext}^n_R(-,N) = R^n\,\mathrm{Hom}(-,N).}$$

Or, for every module M, one has:-

$$\boxed{\mathrm{Ext}^n_R(M,N) = R^n\,\mathrm{Hom}(-,N)(M).}$$

§ To compute $\mathrm{Ext}^n_R(M,N)$:-

Recall that compute right derived functors of a contravariant functor you have to start with a projective resolution:

So, let

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$$

be a projective resolution of M.

Apply $\mathrm{Hom}_R(-, N)$, ~~to get~~ and consider

$$\mathrm{Hom}_R(P_0, N) \longrightarrow \mathrm{Hom}_R(P_1, N) \longrightarrow \mathrm{Hom}_R(P_2, N) \longrightarrow \cdots$$

Take cohomology of this complex to get

$$\boxed{\mathrm{Ext}^n_R(M, N) := H^n(\mathrm{Hom}_R(P_\bullet, N)).}$$

§. Recall this exercise from HW #3 :-

If F is a left exact functor then $R^0F \simeq F$.

Applied to the situation of. $F = \mathrm{Hom}(-, N)$ it gives us:-

$$\boxed{\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N).}$$

§ <u>Long exact Ext-sequence in the first variable</u>:-

The long exact sequence of (right) derived functors applied to Ext^1 gives:-

For any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have the long exact sequence:-

$$\boxed{0 \longrightarrow \mathrm{Hom}(C, N) \longrightarrow \mathrm{Hom}(B, N) \longrightarrow \mathrm{Hom}(A, N) \longrightarrow \mathrm{Ext}^1(C, N) \longrightarrow \mathrm{Ext}^1(B, N) \longrightarrow \cdots}$$

§ A fundamental theorem characterizing projective modules can be proved :-

Theorem

> Let M be an R-module. The following are equivalent
>
> ① M is projective
>
> ② $\mathrm{Ext}^n(M, N) = 0 \quad \forall n \geq 1, \quad \forall$ modules N
>
> ③ $\mathrm{Ext}^1(M, N) = 0 \quad$ ~~$\forall$~~ $\forall$ modules N.

Pf :- ① $\Rightarrow$ ②.

M is projective. So a projective resolution for M may be taken as :-

$$0 \leftarrow M \xleftarrow{1} \underset{P_0}{\overset{\parallel}{M}} \leftarrow \underset{P_1}{\overset{\parallel}{0}} \leftarrow \underset{P_2}{\overset{\parallel}{0}} \cdots$$

$\mathrm{Ext}^n(M,N)$ is the cohomology of the complex :-

$$\underset{\mathrm{Hom}(P_0,N)}{\overset{\parallel}{\mathrm{Hom}(M,N)}} \longrightarrow \underset{\mathrm{Hom}(P_1,N)}{\overset{\parallel}{0}} \longrightarrow 0 \longrightarrow \cdots$$

$\Bigg\} \Rightarrow \mathrm{Ext}^n(M,N) = 0 \quad \forall n \geq 1.$

② $\Rightarrow$ ③ is trivial!

③ $\Rightarrow$ ① $\quad$ If $\mathrm{Ext}^1(M,N) = 0 \quad \forall$ modules N.

Take a projective module P, such that $P \twoheadrightarrow M$ and consider

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0.$$

The associated long exact sequence looks like:

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(P,N) \longrightarrow \mathrm{Hom}(K,N) \longrightarrow \underset{0}{\underset{\parallel}{\mathrm{Ext}^1(M,N)}} \longrightarrow \mathrm{Ext}^1(P,N) \longrightarrow \cdots$$

$\Rightarrow \mathrm{Hom}(P,N) \twoheadrightarrow \mathrm{Hom}(K,N)$.

Take $N = K$. $\Rightarrow$ $1_K \in \mathrm{Hom}(K,K)$ can be lifted to $P \longrightarrow K$.

i.e., the sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ splits.

$\Rightarrow \quad P = K \oplus M$

$\Rightarrow \quad M$ is a summand of a projective $\Rightarrow$ M is projective.

§. Dependence of $\mathrm{Ext}^n(M,N)$ on the second variable N:-

Let $f : N \longrightarrow N'$ be a module homomorphism.

Then f induces a homomorphism $\mathrm{Ext}^n(M,N) \longrightarrow \mathrm{Ext}^n(M,N')$.

because:-

Fix a (proj.) resolution of M: $0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$

Apply $\mathrm{Hom}(-,N)$ and $\mathrm{Hom}(-,N')$ and notice that we have a diagram

$$\begin{tikzcd}
\mathrm{Hom}(P_0,N) \arrow[r] \arrow[d] & \mathrm{Hom}(P_1,N) \arrow[r] \arrow[d] & \mathrm{Hom}(P_2,N) \arrow[r] \arrow[d] & \cdots \\
\mathrm{Hom}(P_0,N') \arrow[r] & \mathrm{Hom}(P_1,N') \arrow[r] & \mathrm{Hom}(P_2,N') \arrow[r] & \cdots
\end{tikzcd}$$

The induced map in cohomology gives $\mathrm{Ext}^n(M,N) \longrightarrow \mathrm{Ext}^n(M,N')$.

The moral is this:-

- $\mathrm{Ext}^n(M,N)$ is a covariant functor in N.
- By definition, $\mathrm{Ext}^n(M,N)$ is a contravariant functor in M.

(One can say that $\mathrm{Ext}^n(M,N)$ is a "bi-functor".)

§ Long exact. Ext-sequence in the second variable:-

The previous discussion of dependence on N in $\mathrm{Ext}^n(M,N)$ can be pushed further to prove:-

Any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives the long exact sequence:-

$$\boxed{0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \to \mathrm{Ext}^1(M,A) \to \mathrm{Ext}^1(M,B) \to \cdots}$$

<u>Proof</u>:- Take a Projective resolution $0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ of M and consider the "grid":-

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(P_0,A) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0,B) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0,C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(P_1,A) \arrow[r] \arrow[d] & \mathrm{Hom}(P_1,B) \arrow[r] \arrow[d] & \mathrm{Hom}(P_1,C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(P_2,A) \arrow[r] \arrow[d] & \mathrm{Hom}(P_2,B) \arrow[r] \arrow[d] & \mathrm{Hom}(P_2,C) \arrow[r] \arrow[d] & 0. \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

observe that each row is <u>exact</u> because P_n is projective.

The associated long exact sequence does the job!

§. The dual theorem characterizing injective modules:-

<u>Theorem</u>

The following are equivalent: conditions on N

① N is injective

② $\mathrm{Ext}^n(M,N) = 0 \quad \forall\, n \geq 1, \ \forall$ modules M.

③ $\mathrm{Ext}^1(M,N) = 0 \quad \forall$ modules M.

§ other points of view.

§§. We could have started the story with the covariant left exact functor given by $\mathrm{Hom}(M,-)$, and considered its right derived functors. (To compute the n^{th} derived functor for N, we would work with an injective resolution $0 \to N \to I^0 \to I^1 \to \cdots$, then we would apply $\mathrm{Hom}(M,-)$ and consider

$$\mathrm{Hom}(M, I^0) \to \mathrm{Hom}(M, I^1) \to \cdots$$

Then $$\mathrm{Ext}^n(M,N) := H^n(\mathrm{Hom}(M, I^\bullet)).$$

It is a fact that this $\mathrm{Ext}^n(M,N)$ is the same as the previous $\mathrm{Ext}^n(M,N)$.

§§. (Yoneda's interpretation).

Let $\mathrm{YExt}^1(M,N)$ be the set of all ~~equivalence~~ short exact sequences

$$0 \to N \to E \to M \to 0.$$

(Such a sequence is called an extension of M by N.)

Two such sequences are considered equivalent if $\exists\, f: E \to E'$ such that

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

Let $\sim$ denote this equivalence.

Let $E^1(M,N) = \mathrm{YExt}^1(M,N)/\sim$

So $E^1(M,N)$ = set of all equivalence classes of extensions of M by N.

Theorem

$$\boxed{\mathrm{Ext}^1_R(M,N) \text{ is in bijection with } E^1(M,N)}$$

Note: (1) This generalizes to higher Ext-groups but the notion of equivalence is a little tricky.