Lecture-8

The TOR - functors.

§1 Fix a right R-module M. So $M \in ob(\text{MOD-}R)$.

Consider the functor $M \otimes_R - : R\text{-MOD} \longrightarrow AB$.

Then $M \otimes_R -$ is a right exact covariant functor.

(Note that it is not in general an exact functor; Here is an example:- $R = \mathbb{Z}$, $M = \mathbb{Z}/2$. Apply $\mathbb{Z}/2 \otimes -$ to the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

to get

$$\begin{array}{ccccc} \mathbb{Z} \otimes \mathbb{Z}/2 & \xrightarrow{2} & \mathbb{Z} \otimes \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \otimes \mathbb{Z}/2 \longrightarrow 0 \\ \wr\wr & & \wr\wr & & \\ \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & & \end{array}$$

So the left-most map is not injective!).

Defn:-
$$\boxed{Tor_n^R(M,N) = (L_n M \otimes_R -)(N).}$$

That is Tor_n^R is the n^{th} - left derived functor of tensoring.

§2 To compute $Tor_n^R(M,N)$. :-

Start with a projective resolution of N

$$0 \leftarrow N \leftarrow Q_0 \leftarrow Q_1 \leftarrow \cdots$$

Apply $M \otimes_R -$, and consider

$$M \otimes_R Q_0 \leftarrow M \otimes_R Q_1 \leftarrow \cdots$$

Take homology, to get

$$\boxed{Tor_n^R(M,N) = H_n(M \otimes_R Q_\bullet).}$$

§3 Long exact sequence in the second variable :-

Any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives a long exact sequence

(Recall this exercise:- If F is right exact then $F \simeq L_0 F$.) Therefore, we get

~~$0 \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow Tor_1^R(M,A) \longrightarrow Tor_1^R(M,B) \longrightarrow$~~

$$\cdots \longrightarrow Tor_1^R(M,C) \longrightarrow Tor_1^R(M,B) \longrightarrow Tor_1^R(M,C) \longrightarrow M \otimes A \longrightarrow M \otimes B \longrightarrow M \otimes C \longrightarrow 0$$

§4 Dependence of $Tor_n^R(M,N)$ on the first variable.

This works out in the same fashion as the Ext-functors.

This works out in the same fashion as the Ext-functors; i.e., it is bi-functor. In particular, it is a functor in both variables ;

Given any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of right R-modules there is a long exact sequence

$$\cdots \longrightarrow Tor_1(A,N) \longrightarrow Tor_1(B,N) \longrightarrow Tor_1(C,N) \longrightarrow A \otimes N \longrightarrow B \otimes N \longrightarrow C \otimes N \longrightarrow 0$$

§5. Flat modules.

Recall :- M is projective $\Leftrightarrow$ Hom (M, –) is exact.

Flat modules is the "tensor product" version of projective modules.

For any M, $M \otimes -$ is always right exact.

Defn :- M is said to be flat if $M \otimes -$ is an exact functor.

Theorem

① M is flat
② $\mathrm{Tor}_n^R(M, N) = 0$ $\forall n \geq 1$, $\forall$ modules N
③ $\mathrm{Tor}_1^R(M, N) = 0$ $\forall$ modules N.

} These are equivalent.

Pf :- This is an exercise. (Look up Jacobson's Basic Alg, vol. II Theorem 6.10)
