Mayer-Vietoris Sequence

This is a long exact sequence in cohomology that relates the cohomology of a union $A \cup B$ in terms of the cohomologies of A, of B and of $A \cap B$. It is a powerful tool in actual computations. We have already used it in computing $H^q(S^n, \mathbb{Z})$!! Later we will compute cohomology of $\mathbb{P}^n(\mathbb{C})$ using this sequence.

§1. Recall some facts about functoriality:

$f: X \longrightarrow Y$ continuous map, $\mathcal{F}$ - sheaf on X, $\mathcal{G}$ - sheaf on Y.

(1) . $f_* F$ - direct image sheaf - it is on Y

$$f_* F(V) = \mathcal{F}(f^{-1}(V)).$$

. $f_* : S_X \longrightarrow S_Y$ is a left exact functor that maps injectives to injectives

. It makes sense to talk of $R^q f_*(\mathcal{F})$ .. etc..

(2) $f^*\mathcal{G}$ - inverse image sheaf = sheafification of the presheaf inverse image which we denote ${}^p f^* \mathcal{G}$.

$$. \quad {}^p f^* \mathcal{G}(U) = \varinjlim_{f(U) \subset V} \mathcal{G}(V).$$

$$. \quad f^*\mathcal{G}_x = ({}^p f^* \mathcal{G})_x = \mathcal{G}_{f(x)}$$

. f^* is an exact functor $f^* : S_Y \longrightarrow S_X$.

③ <u>Adjunction formula</u>

$$\begin{tikzcd} \mathcal{F} \arrow[d] & \mathcal{G} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$$\mathrm{Hom}_{S_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{S_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Adjunction ~~homo~~morphisms :- $\mathcal{G} \longrightarrow f_* f^* \mathcal{G}.$

$$f^* f_* \mathcal{F} \longrightarrow \mathcal{F}.$$

④ <u>Morphisms of functoriality</u> :- $H^q(Y, \mathcal{G}) \longrightarrow H^q(X, f^*\mathcal{G}).$

$$H^q(Y, \mathbb{Z}) \longrightarrow H^q(X, \mathbb{Z}).$$

⑤ <u>Closed embeddings.</u>

Let A be a closed subset of X. , $i : A \hookrightarrow X$

- $i_* : S_A \longrightarrow S_X$ is an exact functor.
- $(i_*\mathcal{G})_x = \begin{cases} \mathcal{G}_x, & x \in A \\ 0, & x \notin A. \end{cases}$ $\forall$ sheaves $\mathcal{G}$ on A.
- $(i_* i^* \mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in A \\ 0, & x \notin A \end{cases}$ $\forall$ sheaves $\mathcal{F}$ on X.
- $H^q(A, \mathcal{G}) = H^q(X, i_*\mathcal{G}).$

⑥ <u>Open embeddings.</u>

Let U be an open subset of X , $j : U \hookrightarrow X$

- $j^*\mathcal{F} = {}^{p}j^*\mathcal{F}$, ${}^{p}j^*\mathcal{F}(V) = \mathcal{F}(V)$ $\forall\, V \underset{\text{open}}{\subseteq} U.$
- $(j_* j^* \mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in U \\ \sim\sim, & x \in \bar{U} - U \\ 0, & x \notin \bar{U} \end{cases}$
- j^* - pulls back flabby resolutions to flabby resolutions.

§2 Mayer-Victoris – I (Closed subsets)

Let A, B be closed subspaces of X.

Let $\mathcal{F}$ be a sheaf on X. If $i : A \hookrightarrow X$ then we use the notation

$$H^q(A, \mathcal{F}) = H^q(A, i^*\mathcal{F}).$$

There is a long exact sequence:

$$\cdots \longrightarrow H^q(A\cup B, \mathcal{F}) \longrightarrow H^q(A,\mathcal{F}) \oplus H^q(B,\mathcal{F}) \longrightarrow H^q(A\cap B, \mathcal{F}) \longrightarrow H^{q+1}(A\cup B, \mathcal{F}) \longrightarrow \cdots$$

<u>Proof</u>:- Let $i : A \hookrightarrow X$, $j : B \hookrightarrow X$ be inclusions.

Let $\mathcal{F}_A = i_* i^* \mathcal{F}$. Similarly $\mathcal{F}_B$, $\mathcal{F}_{A\cup B}$, $\mathcal{F}_{A\cap B}$ make sense

Recall :- $$\mathcal{F}_{A,x} = \begin{cases} \mathcal{F}_x, & x \in A \\ 0, & x \notin A. \end{cases}$$

There is a short exact sequence of sheaves on X:

$$0 \longrightarrow \mathcal{F}_{A\cup B} \longrightarrow \mathcal{F}_A \oplus \mathcal{F}_B \longrightarrow \mathcal{F}_{A\cap B} \longrightarrow 0.$$

Check exactness by checking at the level of stalks: (But before that, the morphisms are:

$\mathcal{F}_{A\cup B} \longrightarrow \mathcal{F}_A \oplus \mathcal{F}_B$ - sum of restrictions

$\mathcal{F}_A \oplus \mathcal{F}_B \longrightarrow \mathcal{F}_{A\cap B}$ - difference of " .)

At the level of stalks:-

$0 \to \mathcal{F}_x \to \mathcal{F}_x \oplus \mathcal{F}_x \to \mathcal{F}_x \to 0$ if $x \in A\cap B$.

$0 \to \mathcal{F}_x \to \mathcal{F}_x \oplus 0 \to 0 \to 0$ if $x \in A \setminus B$

$0 \to \mathcal{F}_x \to 0 \oplus \mathcal{F}_x \to 0 \to 0$ if $x \in B \setminus A$.

Take the associated long exact sequence in cohomology :-

$$\cdots \to H^q(X, \mathcal{F}_{A\cup B}) \to H^q(X, \mathcal{F}_A \oplus \mathcal{F}_B) \to H^q(X, \mathcal{F}_{A\cap B}) \to H^{q+1}(X, \mathcal{F}_{A\cup B}) \to \cdots$$

Finally, observe that

$$\begin{aligned} H^q(X, \mathcal{F}_A) &= H^q(X, i_* i^* \mathcal{F}) \\ &= H^q(A, i^* \mathcal{F}) && \text{(since } i : A \hookrightarrow X \text{ is a closed embedding)} \\ &= H^q(A, \mathcal{F}) && \text{(notation)} \end{aligned}$$

So the above exact sequence looks like

$$\cdots \to H^q(A\cup B, \mathcal{F}) \to H^q(A, \mathcal{F}) \oplus H^q(B, \mathcal{F}) \to H^q(A\cap B, \mathcal{F}) \to H^{q+1}(A\cup B, \mathcal{F}) \to \cdots$$

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§3 Mayer-Vietoris Sequence - II (open subsets)

Let U, V be open subsets of X.
Let $\mathcal{F}$ be a sheaf on X.
Then there is a long exact sequence:

$$\cdots \to H^q(U\cup V, \mathcal{F}) \to H^q(U,\mathcal{F}) \oplus H^q(V,\mathcal{F}) \to H^q(U\cap V, \mathcal{F}) \to H^{q+1}(U\cup V, \mathcal{F}) \to \cdots$$

<u>Proof</u>:- Note:- The sequence looks exactly like it does for the case of closed subsets A & B, however the proof is quite different. The basic problem being that, if $i : U \hookrightarrow X$ then the stalks of $\mathcal{F}_U := i_* i^* \mathcal{F}$ are not easily described for points in $\bar{U} - U$.

$$\mathcal{F}_{U,x} = \begin{cases} \mathcal{F}_x, & x \in U \\ \sim, & x \in \bar{U} - U \\ 0, & x \notin \bar{U} \end{cases}$$

$\longrightarrow$ Depends on the behaviour of $\mathcal{F}$ on the boundary of U.

Let $0 \to \mathcal{F} \to \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots \to \mathcal{J}^n \to \cdots$ be a flabby resolution. Then we claim that i^* pulls this back to a flabby resolution of sheaves on U.

<u>check</u>:- • i^* is exact

• $\mathcal{J}$ is flabby $\Rightarrow$ $i^*\mathcal{J}$ is flabby: $V \subset U$ open

$$\begin{array}{ccc} i^*\mathcal{J}(U) & \longrightarrow & i^*\mathcal{J}(V) \\ \| & & \| \\ \mathcal{J}(U) & \twoheadrightarrow & \mathcal{J}(V) \end{array}$$

Hence

$$0 \to i^*\mathcal{F} \to i^*\mathcal{J}^0 \to i^*\mathcal{J}^1 \to \cdots \to i^*\mathcal{J}^n \to \cdots$$

is a flabby resolution of $i^*\mathcal{F}$. In particular,

$$\begin{aligned} H^q(U, \mathcal{F}) &= H^q(U, i^*\mathcal{F}) \\ &= H^q(i^*\mathcal{J}^\bullet(U)) \\ &= H^q(\mathcal{J}^\bullet(U)) \end{aligned}$$

$i^*\mathcal{J}(U) = \mathcal{J}(U)$.

Same comments apply to $H^q(V, \mathcal{F})$, $H^q(U\cup V, \mathcal{F})$, $H^q(U\cap V, \mathcal{F})$.

Next, observe that if $\mathcal{J}$ is a flabby sheaf on X then one has the exact sequence :

$$0 \to \mathcal{J}(U \cup V) \longrightarrow \mathcal{J}(U) \oplus \mathcal{J}(V) \longrightarrow \mathcal{J}(U \cap V) \longrightarrow 0.$$

Now consider the complex :-

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{J}^0(U \cup V) \arrow[r] \arrow[d] & \mathcal{J}^0(U) \oplus \mathcal{J}^0(V) \arrow[r] \arrow[d] & \mathcal{J}^0(U \cap V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}^1(U \cup V) \arrow[r] \arrow[d] & \mathcal{J}^1(U) \oplus \mathcal{J}^1(V) \arrow[r] \arrow[d] & \mathcal{J}^1(U \cap V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}^2(U \cup V) \arrow[r] \arrow[d] & \mathcal{J}^2(U) \oplus \mathcal{J}^2(V) \arrow[r] \arrow[d] & \mathcal{J}^2(U \cap V) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

This is a short exact sequence of complexes.

Take the associated long exact sequence in cohomology: to get :-

$$\cdots \to H^q(\mathcal{J}^\bullet(U \cup V)) \longrightarrow H^q(\mathcal{J}^\bullet(U) \oplus \mathcal{J}^\bullet(V)) \longrightarrow H^q(\mathcal{J}^\bullet(U \cap V)) \longrightarrow H^{q+1}(\mathcal{J}^\bullet(U \cup V)) \to \cdots$$

$$H^q(\mathcal{J}^\bullet(U) \oplus \mathcal{J}^\bullet(V)) = H^q(\mathcal{J}^\bullet(U)) \oplus H^q(\mathcal{J}^\bullet(V))$$

and use $H^q(\mathcal{J}^\bullet(U)) = H^q(U, \mathcal{F})$. to get

$$\cdots \to H^q(U \cup V, \mathcal{F}) \to H^q(U, \mathcal{F}) \oplus H^q(V, \mathcal{F}) \longrightarrow H^q(U \cap V, \mathcal{F}) \longrightarrow H^{q+1}(U \cup V, \mathcal{F}) \longrightarrow \cdots$$

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