Math 5293, Spring 2007 Homework Problems

[ ]'s indicate comprehensive exam dates, if applicable. Also D denotes the open unit disk.

1. [Aug. 06] Let $\mathcal{F}$ be the family of all functions f holomorphic in D such that f has a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with $|c_n| \le n^2$ for all n. Show that $\mathcal{F}$ is a normal family.

2. [Jan. 98 & Jan. 03] Let $\mathcal{F}$ denote the family of functions f analytic on D so that $f(0) = 0$ and

$$|f'(z)| \le \frac{1}{1-|z|}$$

for all $z \in D$. Show that $\mathcal{F}$ is a normal family.

3. [Aug. 03] Let $\mathcal{F}$ denote the family of functions f for which there exists an open set containing the closure of D on which f is analytic and so that

$$\int_0^{2\pi} |f(e^{it})|dt \le 1.$$

Show that $\mathcal{F}$ is a normal family in D.

4. [Jun. 98 & Aug. 99] Let $\mathcal{F}$ denote the set of all f analytic on D such that

$$\iint_D |f(x+iy)|^2 dxdy \le 1.$$

Show that $\mathcal{F}$ is a normal family. (Hint: Fix $p \in D$ and show first that

$$|f(p)|^2 \le \frac{1}{2\pi} \int_0^{2\pi} |f(p + re^{it}|^2 dt$$

for $0 < r < 1 - |p|$. Then ...)

5. [Aug. 00 & Aug. 99] By definition, an *automorphism* of a connected open set G is a one-to-one analytic function mapping G onto G.
(a) Let f and g be automorphisms of D. Suppose that there are two distinct points P and Q in D such that $f(P) = g(P)$ and $f(Q) = g(Q)$. Show that $f = g$ on D.
(b) Let Ω be a simply connected region in $\mathbb{C}$ which is not equal to $\mathbb{C}$, and let f and g be automorphisms of Ω. Suppose that there are two distinct points P and Q in Ω such that $f(P) = g(P)$ and $f(Q) = g(Q)$. Show that $f = g$ on Ω.

6. [Aug. 96] Let $\Omega = \{x + iy : x, y \in \mathbb{R},\, x^2 < y^2 + 1\}$ and let f be an entire function so that $f(\mathbb{C}) \subset \Omega$. Show that f is constant.

7. [Jun. 98; also look at # 1 of Jan. 97 and # 4 of Aug. 96] (a) Suppose that u is a real-valued harmonic function in D. Show that u is the real part of some function f analytic in D.
(b) Define $D' = D \setminus \{0\}$. Let $u(z) = \log(|z|)$. Show that u is harmonic in D', but it is *not* the real part of any function analytic in D'.

8. [Aug. 04] (a) Let u be a continuous function on the closure of D which is harmonic on D. Express $u(0)$ in terms of an integral along the unit circle.
(b) Let u be as in part (a). If $0 < r < 1$ and θ is real, give a formula for $u(re^{i\theta})$ in terms of an integral along the unit circle.
(c) Let ϕ be the function defined on the unit circle by $\phi(e^{it}) = 1$ if $\alpha \le t \le \beta$ and $\phi = 0$ otherwise. Here α and β are constants with $0 \le \alpha < \beta \le 2\pi$. Using ϕ in place of u in the integral you found in part (b) defines a function on D. What is the value of that function at 0?

9. [Ahlfors P. 166] (Hadamard's Three-Circle Theorem) Suppose that $f(z)$ is analytic in the annulus $r_1 < |z| < r_2$ and continuous on the closed annulus. If $M(r)$ denotes the maximum of $|f(z)|$ for

$|z| = r$, show that

$$M(r) \leq M(r_1)^{\alpha} M(r_2)^{1-\alpha},$$

where $\alpha = \frac{\log(r_2/r)}{\log(r_2/r_1)}$. When does equality hold?

10. [Ahlfors P. 171, Problems 6,7] (a) If $f(z)$ is entire and $z^{-1}Ref(z) \to 0$ as $z \to \infty$, show that f is a constant.
(b) If $f(z)$ is holomorphic in a neighborhood of ∞ and $z^{-1}Ref(z) \to 0$ as $z \to \infty$, show that $\lim_{z\to\infty} f(z)$ exists.

11. [May. 96] (a) Suppose that f is analytic in a region G such that its real part depends only on $|z|$ but not arg(z). Show that its real part is necessarily of the form $A \log |z| + B$ for some constants A and B. Is it possible for such a function to exist?
(b) Suppose that $\{u_n\}$ is a sequence of non-negative functions harmonic on $D(0,2) = \{z : |z| < 2\}$ such that $u_n(0) \to \infty$. Show that $u_n(1) \to \infty$.

12. [Jan. 04] (a) Give an example of a bounded harmonic function on D whose harmonic conjugate is unbounded.
(b) Does there exist a nonconstant bounded harmonic function on $\mathbb{C}$?

13. [Jan. 03] Let Ω be a connected open set in $\mathbb{C}$.
(a) For which functions f analytic on Ω is $|f|^2$ harmonic on Ω?
(b) Show that if f is analytic and nonzero on Ω, then $\log |f|$ is harmonic on Ω.

14. [Jan. 97] (a) Show that any continuous real-valued function $u(x, y)$ which satisfies the Mean Value Property in a region Ω is harmonic in Ω.
(b) Use (a) to prove the Schwarz Reflection Principle for harmonic functions in the following simple form: If $u(x, y)$ is a continuous real-valued function on the half disk $\{z : |z| < 1 \,\text{and}\, \text{Im(z)} > 0\}$ and $u(x, y) \equiv 0$ on $\{z : |z| < 1 \text{ and } \text{Im(z)} = 0\}$, then $u(x, y)$ has a harmonic extension to all of D.

15. [Jan. 97] Let S be the open sector defined by $S = \{z \in \mathbb{C} : 0 < |z| < 1 \text{ and } 0 < \arg(z) < \pi/2\}$. Is this set conformally equivalent to the unit disk D? If so, explicitly determine a conformal equivalence and determine the image of the intersection of S with the ray $R = \{z : \arg z = \pi/4\}$. If not, prove why not.

16. [Ahlfors, P. 174, # 3] If $f(z)$ is analytic in $|z| \leq 1$ and satisfies $|f| = 1$ on $|z| = 1$, then show that $f(z)$ is rational. (Note: You already saw this problem last semester on the mid-term exam. The proof then used some elementary arguments and we even proved what form the function $f(z)$ should have. Now, we want a more sophisticated and hopefully faster proof.)

17. [Ahlfors, P. 238, # 6] Determine a conformal mapping of the upper half plane onto the region $\Omega = \{z = x + iy : x > 0, y > 0, \min(x, y) < 1\}$.

18. [Ahlfors, P. 248, #3] If $v(z)$ is continuous together with its partial derivatives up to the second order, prove that v is subharmonic if and only if $\Delta v \geq 0$.

19. [Ahlfors, P. 200] What are the residues of $\Gamma(z)$ at the poles $z = -n$?

20. [Aug. 03] Suppose that f is an entire function such that $f(0) = f(\pi) = 0$. Further assume that for all $z \in \mathbb{C}$ we have $f(z + 2\pi) = f(z)$ and $|f(z)| \leq \exp |\text{Im } z|$. Show that there exists a constant c such that $f(z) = c \sin(z)$ for all $z \in \mathbb{C}$.

21. [Ahlfors, P. 206] For real $x > 0$ prove that

$$\Gamma(x) = \sqrt{2}x^{x-1/2}e^{-x}e^{\theta(x)/12x}$$

with $0 < \theta(x) < 1$.

22. [Mellin Inversion Formula] Show that for real $x > 0$ we have
$$e^{-x} = \frac{1}{2\pi i} \int_{\sigma=\sigma_0} x^{-s}\Gamma(s)ds,$$
where $s = \sigma + it$, and the integral is taken on a vertical line with fixed real part $\sigma_0 > 0$, and $-\infty < t < \infty$.

23. Show that
$$\zeta(s) = \frac{1}{s-1} + \phi(s),$$
where $\phi(s)$ is holomorphic for Re $(s) > 0$. In particular, this implies that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.

24. Let $\Omega = \{z \in \mathbb{C} : \text{Re } (z) > 0\}$ be the open right half plane. Suppose that $f(z)$ is a continuous function on the closure of Ω, holomorphic in Ω, and satisfies
$$|f(z)| < A\exp(|z|^\alpha), \quad z \in \Omega$$
for constansts $A < \infty$ and $\alpha < 1$. Furthermore, assume that
$$|f(iy)| \leq 1$$
for all real y. Prove that $|f(z)| \leq 1$ in Ω. Also, show that the conclusion is false for $\alpha = 1$.

Due: Last Meeting before the Final

25. [Ahlfors, P. 274, #1] Show that any even elliptic function with periods ω_1 and ω_2 can be expressed in the form
$$C\prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)} \quad (C \text{ constant}),$$
provided that 0 is neither a zero nor a pole.

26. [Ahlfors, P. 275, #2] Show that any elliptic function with periods ω_1 and ω_2 can be written as
$$C\prod_{k=1}^{n} \frac{\sigma(z - a_k)}{\sigma(z - b_k)} \quad (C \text{ constant}).$$

27. [Ahlfors, P. 282] Show that the function
$$J(\tau) = \frac{4}{27}\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}$$
is automorphic with respect to the full modular group. Where does it take the values 0 and 1, and with what multiplicities? Here, $\lambda = \lambda(\tau)$ is the mapping we introduced in class.

28. [Ahlfors, P. 290, #1] If a function element is defined by a power series inside its circle of convergence, supposed to be of finite radius, prove that at least one radius is a singular path for the global analytic function which it determines.

29. [Ahlfors, P. 291, #2] If a function element (f, Ω) has no direct analytic continuation other than the ones obtained by restricting f to a smaller region, then the boundary of Ω is called *a natural boundary* for f. Prove that the series $\sum_{n=0}^{\infty} z^{n!}$ has the unit circle as a natural boundary.

30. [Ahlfors, P. 291, #3] Show that the function $\lambda(\tau)$ introduced in class has the real axis as a natural boundary.