Weil Conjecture

$X \subseteq \mathbb{P}_N(\mathbb{C})$ non-sing. m-dim proj variety

Suppose X eq $f_1, \dots, f_k = 0$

$f_i \in R[x_0 \dots x_N]$
↑ alg number ring ($\mathbb{Z}$)

π max ideal in R ($p\mathbb{Z}$) $\to \mathbb{F}_q = R/\pi$

$X_\pi \subseteq \mathbb{P}_N(\mathbb{F}_q) \leftarrow$ "reduce mod q"

Prop: X_π is smooth over $\mathbb{F}_q$ for all but finite # of primes p

Defn: $\overline{X}_\pi$ corresp. variety over $\overline{\mathbb{F}}_q$ (alg closure)

$\forall r \geq 1$, $\exists! \, \mathbb{F}_{q^r} \subseteq \overline{\mathbb{F}}_q$

$\uparrow$ solns to $x^{q^r} - x = 0$

In fact $\overline{\mathbb{F}_q} = \bigcup_{r \geq 1} \mathbb{F}_{q^r}$

and $\mathbb{F}_{q^r} \subseteq \mathbb{F}_{q^t}$ iff $r | t$.

Q: How many pts $(x_0, \dots, x_N) \in \overline{X}_\pi$ s.t.

$x_j \in \mathbb{F}_{q^r}$ $\forall j$?

Let N_r be this number.

Ex) $X = \mathbb{P}_m$

$N_r = \#(x_0, \dots, x_m)$ homog. coordinates
$x_j \in \mathbb{F}_{q^r}$

$\longrightarrow N_r = (q^r)^{m+1} - 1 \Big/ (q^r - 1)$

$= q^{mr} + \cdots + q^r + 1$

Zeta fn: $Z(t) = \exp\left(\sum_{r \geq 1} \frac{N_r}{r} \cdot t^r\right)$

Ex) $X = \mathbb{P}_m$

$$Z(t) = \exp\left(\sum_{r \geq 1} (1 + q^r + q^{2r} + \cdots + q^{mr}) \cdot t^r/r\right)$$

$$= \exp\left(\sum_{k=0}^{m} \sum_{r \geq 1} \frac{1}{r} (q^k t)^r\right)$$

$$= \exp\left(\sum_{k=0}^{m} \log\left((1 - q^k t)^{-1}\right)\right)$$

$$= \frac{1}{(1-t)(1-qt)(1-q^2t)\cdots(1-q^m t)}$$

Weil Conjectures

① $Z(t) = \dfrac{P_1(t)\, P_3(t) \cdots P_{2m-1}(t)}{P_0(t)\, P_2(t) \cdots P_{2m}(t)}$

s.t. ⓐ $P_0(t) = 1 - t$

ⓑ $P_{2m}(t) = 1 - q^m t$

rationality ③ $P_j(t) \in \mathbb{Z}[t]$.

$$P_j(t) = \prod_{1 \le i \le \beta_j} (1 - \alpha_{ji} t)$$

$$\beta_j = \dim H_j(X)$$

$$|\alpha_{ji}| = q^{j/2} \leftarrow \text{R. hyp.}$$

② Let $E = \sum_j (-1)^j \beta_j$ (Euler char)

then

$$Z\left(1/(q^m t)\right) = \pm q^{mE/2} t^E Z(t)$$

Remark: $Z(t)$ uniquely determines $P_j(t)$

N_r's determine Betti numbers β_j's !

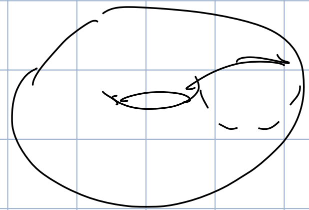
Ex) Curves ($\dim_{\mathbb{C}} X = 1$)

① $\dim(H_i(X)) = 1$ for $i = 0, 2$

② $\dim(H_1(X)) = 2g$ g = genus

N_r's determine g !

($g=1$) Elliptic curves



$$Z(t) = \frac{1 + at + qt^2}{(1-t)(1-qt)}$$

$a \in \mathbb{Z}$, $|a| \leq 2\sqrt{q}$

Weil proved his conjecture for smooth curves (1949)

Q: How does homology fit in?

Lefschetz (-Hopf) fix pt formula

Let X be a cpt, triangulable-space

Suppose $f: X \to X$ cont. with

finite # of fix pts (with multi).

$$\# \text{fix}(f) = \sum_{k \geq 0} (-1)^k \, tr\left(f_* \big|_{H_k(X)}\right)$$

Q: Can we use this to count N_r?

Frobenius map: $f : \overline{X}_{\pi} \to \overline{X}_{\pi}$

$$f(x_0, \dots, x_N) = (x_0^q, \dots, x_N^q)$$

Ex) $x^2 + xy = 0 \to (x^q)^2 + x^q y^q$

in $\mathbb{F}_q \longrightarrow$
$$= (x^2)^q + (xy)^q$$
$$= (x^2 + xy)^q = 0$$

← u.g. dream

<u>Key observation</u>:

$x \in \overline{X}_{\pi}$ is a fixed pt of f^r

$\longleftrightarrow$

x has coordinates in $\mathbb{F}_{q^r}$

$x_j^{q^r} = x_j \longleftrightarrow x_j^{q^r} - x_j = 0$

Apply Lefschetz - Hopf to Frobenius map!

Problem: $\overline{X}_{\pi}$ is not a cpt manifold over $\mathbb{R}$

countable set.

Soln: Need a new cohomology theory

ℓ-adic cohomology (Grothendieck)

ℓ prime $\neq p$. $(q = p^s)$

$$\mathbb{Z}_\ell = \varprojlim_r \mathbb{Z}/\ell^r\mathbb{Z}$$

$\mathbb{Q}_\ell$ field of fractions.

$$H^*(X, \mathbb{Q}_\ell) := H^*_{et}(X, \mathbb{Q}_\ell)$$

① "Čech cohomology" of constant sheaf $\mathbb{Q}_\ell$

② étale "topology"

Properties of $H^*(X, \mathbb{Q}_\ell)$:

① $\dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) = \dim_{\mathbb{Q}_\ell} H^i(X, \mathbb{Q}_\ell)$

② If X sm, Poincaré Duality

$$H^i \otimes H^{2m-i} \to H^{2m} \simeq \mathbb{Q}_\ell. \quad (\text{non degen})$$

③ $H^i(X, \mathbb{Q}_\ell) \simeq H^i(\overline{X}_{\pi}, \mathbb{Q}_\ell)$

④ Lefschetz fix pts holds for $f: \overline{X}_{\pi} \to \overline{X}_{\pi}$.

$\leadsto$ trace formula for N_r

<u>Now</u>: $Z(t) = \exp\left(\sum_{r \geq 1} N_r \cdot \frac{t^r}{r}\right)$

$$= \exp\left(\sum_{r \geq 1}\left(\sum_{j=0}^{2m} (-1)^j \operatorname{Tr} (f^r)^* \Big|_{H_j} \frac{t^r}{r}\right)\right)$$

$$= \prod_{j=0}^{2m} \exp\left(\sum_{r \geq 1} \operatorname{Tr}(f^*)^r \Big|_{H^j} \frac{t^r}{r}\right)^{(-1)^j}$$

$\Big($ Let $A_j := f^* \big|_{H^j}$

$$\downarrow$$

$$\prod_{j=0}^{2m} \det\left(\exp\left(\sum_{r\geq 1}\frac{1}{r}(tA_j)^r\right)\right)^{(-1)^j}$$

$$= \prod_{j=0}^{2m} \det(1-tA_j)^{(-1)^{j+1}}$$

$$= \frac{P_1 \cdots P_{2m-1}}{P_0 \cdots P_{2m}} \qquad P_j = \det(1-tA_j)$$

So $P_j = \prod_{i=1}^{\dim H^j}(1-\alpha_{ij}t)$

α_{ij} e.values of $A_j = f^*: H^j \to H^j$

<u>Weil conjectures</u>

① $P_j \in \mathbb{Z}[t] \Leftarrow A_j$ "integral matrix"

($Z(t)$ rational)

② eigenvalues: $|\alpha_{ij}| = q^{j/2}$ (R. hyp.)

$$\begin{tikzcd} \sigma \text{ 2k-cycle} \arrow[d] & \subseteq & \bar{X}_\pi \arrow[d] & \subseteq & \mathbb{P}^N \arrow[d, squiggly] \\ (x_0,\dots,x_k,0,\dots,0) & \subseteq & (x_0,\dots,x_m,0\dots0) & \subseteq & (x_0,\dots,x_N) \end{tikzcd}$$

If σ is an e.vector of f^*, then

$$f^*(\sigma) = c \cdot \sigma$$

how many copies?

$$(x_0, \dots, x_k, 0, \dots, 0) \xrightarrow{f}$$

$$(a_0, \dots, a_k, 0, \dots, 0) = (x_0^q, \dots, x_k^q, 0, \dots, 0)$$

$$x_0^q - a_0 = 0,\ x_1^q - a_1 = 0,\ \dots\dots,\ x_k^q - a_k = 0$$

$$|f^{-1}(\bar{a})| = q^k \to |c| = q^k.$$

for $\sigma \in H_{2k}(X)$

② $Z(1/q^m t) = \pm q^{mE/2} t^E Z(t)$.

follows from Poincaré duality.

Q: What if X is singular?

Weil conjectures fail using $H^*(X, \mathbb{Q}_\ell)$

Fix? replace with $IH^*(X, \mathbb{Q}_\ell)$

① Strata need to be smooth mod π.

② use Deligne's construction

③ $A_j = f^*|_{IH_j}$

Showing e. values $|\alpha_{ji}| = q^{j/2}$

requires

using P. duality.