LECTURE 7

Exact Solutions of First Order ODEs

1. The Fundamental Theorem of Calculus

Consider a first order differential equation of the simple form

$$\frac{dy}{dx} = g(x)$$

To write down the solution of this equation, it suffices to simply apply the Fundamental Theorem of Calculus. Now roughly speaking the Fundamental Theorem of Calculus says that integrals and derivatives are inverses of each other. Here is a more precise statement

THEOREM 7.1 (Fundamental Theorem of Calculus). I. *If f is a continuous function on an interval $[a,b] \subset \mathbb{R}$, that is differentiable on (a,b), and if f' is integrable on $[a,b]$, then*

$$\int_a^b f' dx = f(b) - f(a)$$

II. *Let f be a continuous function on $[a,b]$. For $x \in [a,b]$, let*

$$F(x) \equiv \int_a^x f(x)\, dx$$

Then $F(x)$ is continuous and differentiable on (a,b) and

$$\frac{dF}{dx}(x) = f(x)$$

Now consider the differential equation

$$\frac{dy}{dx} = g(x)$$

let

$$Y(x) \equiv \int_a^x g(x)\, dx$$

then according to Part II of the F.T.o.C. we have

$$\frac{dY}{dx} = g(x)$$

So any function of the form $Y(x) = \int_a^x g(x)\, dx$ will be a solution of $\frac{dy}{dx} = g(x)$.

We'll now restate this observation in a manner more in keeping with the language we'll be using in this course on differential equations. But first, observe that if we set

$$y(x) = \int_a^x g(x)\, dx + C$$

with C a constant, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_a^x g(x)\, dx + C\right) = g(x) + \frac{dC}{dx} = g(x) + 0 = g(x)$$

So functions of the form

$$\int_a^x g(x)\,dx + C$$

are also solution.

THEOREM 7.2. *The general solution solution of*

$$\frac{dy}{dx} = g(x)$$

is

$$y(x) = \int g(x)\,dx + C$$

2. Separation of Variables

Suppose that a first order differential equation

$$y' = F(x,y) \tag{7.1}$$

can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0 \quad . \tag{7.2}$$

Note that the first term depends only on x and the second term depends only on y and y'. In such a case, we say that the differential equation (7.2) is **separable**. Such differential equations can always be solved (at least implicitly).

To construct a solution of (7.2) we rewrite (7.2) as

$$M(x)dx = -N(y)dy \quad .$$

Integrating both sides of this relation yields

$$\int M(x)dx = -\int N(y)dy + C \quad . \tag{7.3}$$

The constant C is an arbitrary constant of integration. This equation can be used to establish y as an implicit function of x.

To see this, let us define functions $H_1(x)$ and $H_2(y)$ by

$$\begin{aligned} H_1(x) &= \int^x M(x')\,dx' \quad , \\ H_2(y) &= \int^y N(y')\,dy' \quad . \end{aligned}$$

Equation (7.3) is now equivalent to

$$H_1(x) = -H_2(y) + C \quad . \tag{7.4}$$

But now equation (7.4) expresses a purely algebraic relation between x and y. Solving (7.4) for y will then give us y as an explicit function of x.

Below is an argument that is a little more rigorous. If we set

$$\begin{aligned} H_1(x) &= \textstyle\int^x M(x')\,dx' \quad , \\ H_2(y) &= \textstyle\int^y N(y')\,dy' \end{aligned}$$

Then by the Fundamental Theorem of Calculus we have

$$\begin{aligned} \frac{dH_1}{dx} &= M(x) \quad , \\ \frac{dH_2}{dy} &= N(y) \quad . \end{aligned}$$

and so

$$M(x) + N(y)y' = 0$$

can be written

$$\begin{aligned} 0 &= \tfrac{dH_1}{dx} + \tfrac{dH_2}{dy}\tfrac{dy}{dx} \\ &= \tfrac{dH_1}{dx} + \tfrac{d}{dx}H_2\left(y(x)\right) \\ &= \tfrac{d}{dx}\left(H_1 + H_2\left(y(x)\right)\right) \end{aligned}$$

(Note that we have employed the "chain rule" in the second step.) But

$$\frac{d}{dx}\left(H_1(x) + H_2\left(y(x)\right)\right) = 0$$

implies that

$$H_1(x) + H_2\left(y\right) = C \quad , \quad \text{(some constant)}.$$

Solving this equation for y as a function of x and C will thus furnish us with a solution of (7.2).

In summary, the general solution of a nonlinear differential equation of the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

is constructed by first computing anti-derivatives $H_1(x)$, $H_2(y)$ of the functions the functions $M(x)$ and $N(y)$;

$$\begin{aligned} H_1(x) &= \textstyle\int^x M(x')\,dx' \\ H_2(y) &= \textstyle\int^y N(y')dy' \end{aligned}$$

and solving the equation

$$H_1(x) + H_2(y) = C$$

for y.

Example 7.3.

(7.5)
$$y' = \frac{y^2}{x}$$

After multiplying both sides by $\frac{x}{y^2}$, this equation can also be rewritten as

$$\frac{1}{x} = \frac{1}{y^2}\frac{dy}{dx} \quad ;$$

or

$$\frac{dx}{x} = \frac{dy}{y^2} \quad .$$

Integrating the left hand side with respect to x and the right hand side with respect to y yields

$$\ln|x| = \int\frac{dx}{x} = \int\frac{dy}{y^2} + C = -\frac{1}{y} + C$$

or

$$\frac{1}{y} = -\ln|x| + C$$

or

$$y(x) = \frac{1}{C - \ln|x|} \quad .$$

The equation above represents the general solution of (7.5). ∎