Math 3013
SOLUTIONS TO FIRST EXAM
February 22, 2008

1. (5 pts each) Give the precise definitions of the following mathematical notions:

(a) the **span** of a set of vectors

The span of a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is the set

$$span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

(b) a **subspace** of $\mathbb{R}^n$

A subspace of $\mathbb{R}^n$ is a subset W of $\mathbb{R}^n$ such that

(i) for all $\lambda \in \mathbb{R}$ and all $\mathbf{w} \in W$, $(\lambda\mathbf{w}) \in W$

(ii) for all $\mathbf{w}, \mathbf{w}' \in W$, $(\mathbf{w} + \mathbf{w}') \in W$

(c) a **basis**

A basis for a subspace W of $\mathbb{R}^n$ is a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ such that each vector in W can be uniquely expressed in the form

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

2. Consider the following linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 + x_4 &= 1 \\ x_2 - x_3 &= -1 \end{aligned}$$

(a) (10 pts) Write down the corresponding augmented matrix and reduce it to **reduced** row-echelon form.

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 2 \\ 2 & -1 & 1 & 0 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) (5 pts) Write down the solution of the original linear system.

Note that column 3 is the only column on the left that doesn't have a pivot. So there'll be one free parameter, x_3, in the solution. The equations corresponding to the reduced row echelon form are

$$\left.\begin{aligned} x_1 &= 1 \\ x_2 - x_3 &= -1 \\ x_4 &= 1 \\ 0 &= 0 \end{aligned}\right\} \implies \mathbf{x} = \begin{bmatrix} 1 \\ -1 + x_3 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

3. (15 pts) For each of the following augmented matrices, describe the solution space of the corresponding linear system. (Is there a solution? If so, is it unique? How many free parameters are needed to describe the general solution?)

(a) $\left[\begin{array}{cccc|c} 3 & 0 & 1 & 3 & 1 \\ 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$ consistent, no colums without pivots $\implies$ unique solution

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ consistent, 1 column without pivot $\implies$ 1-parameter family of solutions

(c) $\left[\begin{array}{cccc|c} 3 & 0 & 1 & 2 & 2 \\ 0 & 3 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ inconsistent (equation corresponding to last row says $0 = 1$).

4. (10 pts) Compute the inverse of

$$\mathbf{A} = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \implies \mathbf{A}^{-1} = \left[\begin{array}{ccc} -1 & -1 & 1 \\ 2 & 3 & -2 \\ 0 & -2 & 1 \end{array}\right]$$

5. (5 pts) Solve

$$\left[\begin{array}{cc} 1 & 1 \\ 3 & 4 \end{array}\right]\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$$

Let $\mathbf{A} = \left[\begin{array}{cc} 1 & 1 \\ 3 & 4 \end{array}\right]$, $\mathbf{X} = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$, $\mathbf{B} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 1 & 0 & 4 & -1 \\ 0 & 1 & -3 & 1 \end{array}\right] \implies \mathbf{A}^{-1} = \left[\begin{array}{cc} 4 & -1 \\ -3 & 1 \end{array}\right]$$

Then multiplying both sides of the matrix equation $\mathbf{AX} = \mathbf{B}$ from the left by $\mathbf{A}^{-1}$ yields

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B} \implies \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \left[\begin{array}{cc} 4 & -1 \\ -3 & 1 \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] = \left[\begin{array}{cc} -1 & 4 \\ 1 & -3 \end{array}\right]$$

So

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} -1 & 4 \\ 1 & -3 \end{array}\right]$$

6. (10 pts) Prove or disprove that the set $W = \{[x, y] \in \mathbb{R}^2 \mid y = 2\}$ is closed under scalar multiplication.

W is not closed under scalar multiplication because $[1, 2] \in W$ by $3 \cdot [1, 2] = [3, 6] \notin W$ since the second component is no longer equal to 2.

7. (10 pts) Prove or disprove that the set $W = \{[x, y] \in \mathbb{R}^2 \mid x + y = 0\}$ is closed under vector addition.

Let $[x, y]$, $[x', y'] \in W$. Then

$$[x, y] + [x', y'] = [x + x', y + y']$$

This lies in W since the sum of its components is 0:

$$(x + x') + (y + y') = (x + y) + (x' + y') = 0 + 0 = 0$$

8. (10 pts) Find a basis for the solution set of the following homogeneous linear system.

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + 3x_3 + x_4 &= 0 \\ -x_2 + 2x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 5 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Longrightarrow \left.\begin{aligned} x_1 + 5x_3 + x_4 &= 0 \\ x_2 - 2x_3 &= 0 \\ 0 &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} &x_1 = -5x_3 - x_4 \\ &x_2 = -2x_3 \\ &x_3, x_4 \text{ free} \end{aligned} \Longrightarrow \mathbf{x} = \begin{bmatrix} -5x_3 - x_4 \\ -2x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\mathbf{x} = x_3 \begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

9. (10 pts) Let $\mathbf{v}_1 = [1, 2, 1, -1]$, $\mathbf{v}_2 = [1, 1, 1, 0]$, $\mathbf{v}_3 = [1, 1, -1, -2]$, $\mathbf{v}_4 = [0, 0, 1, 1]$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $W = span\,(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \quad \text{basis} = \{[1, 0, 0, 0], [0, 1, 0, -1], [0, 0, 1, 1]\}$$