Math 3013

SOLUTIONS TO SECOND EXAM

10:30 – 11:20 am, April 4, 2008

1. Define, precisely, the following notions (where V, W are to be regarded as general vector spaces). space V).

(a) (3 pts) a **subspace** of V

A subspace of a vector space V is a subset W of V such that

- for all $\lambda \in \mathbb{R}$ and all $\mathbf{w} \in w$, $\lambda \mathbf{w} \in W$
- for all $\mathbf{v}, \mathbf{u} \in W$, $\mathbf{v} + \mathbf{u} \in W$

(b) (3 pts) a **basis** for a vector space V

A basis for a vector space V is a set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \in V$ such that every vector $\mathbf{v} \in V$ can be uniquely expressed as

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots .c_n\mathbf{b}_n \qquad , \quad c_1, \ldots, c_n \in \mathbb{R}$$

(c) (3 pts) a **set of linearly independent vectors** in V

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set if the only solution of

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

is $c_1 = 0$, $c_2 = 0$, ... , $c_k = 0$.

(d) (3 pts) a **linear transformation** from a vector space V to vector space W.

A linear transformation from a vector space V to a vector space W is a function $T : V \to W$ such that

- $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$ for all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in V$
- $T(\mathbf{v} + \mathbf{u}) = T(\mathbf{v}) + T(\mathbf{u})$ for all $\mathbf{v}, \mathbf{u} \in V$.

(e) (3 pts) the **null space** of an $n \times m$ matrix $\mathbf{A}$.

The null space of a matrix $\mathbf{A}$ is the solution set of $\mathbf{Ax} = \mathbf{0}$.

2. Consider the following matrix: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{\text{R.R.E.F}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \equiv \mathbf{A}'$$

(a) (5 pts) Find a basis for the row space of $\mathbf{A}$.

A basis for the row space of $\mathbf{A}$ is given by the non-zero rows of the (reduced) row echelon form $\mathbf{A}'$

$$RowSp(\mathbf{A}) = span([1,0,0,2], [0,0,1,3])$$

(b) (5 pts) Find a basis for the column space of $\mathbf{A}$.

A basis for the column space of $\mathbf{A}$ is given by the columns of $\mathbf{A}$ corresponding to the columns of $\mathbf{A}'$ that contain pivots; i.e., columns 1 and 3:

$$ColSp(\mathbf{A}) = span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$

(c) (5 pts) Find a basis for the null space of $\mathbf{A}$.

The general solution of $\mathbf{Ax} = \mathbf{0}$ can be read off the reduced row echelon form $\mathbf{A}'$:

$$\left.\begin{array}{r} x_1 = -2x_4 \\ x_2 = free \\ x_3 = -3x_4 \\ x_4 = free \end{array}\right\} \implies \mathbf{x} = \begin{bmatrix} -2x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix} = span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}\right)$$

(d) (5 pts) What is the rank of $\mathbf{A}$? The rank of $\mathbf{A}$ is the dimension of its row space (or column space)

$$rank(\mathbf{A}) = 2$$

3. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_1 - 2x_3, 2x_1 - 4x_3]$.
(a) Find the matrix $\mathbf{A}_T$ representating T.

$$\left.\begin{array}{ll} T([1,0,0]) & = [1,2] \\ T([0,1,0]) & = [0,0] \\ T([0,0,1]) & = [-2,-4] \end{array}\right\} \implies \mathbf{A}_T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & -4 \end{bmatrix}$$

(b) Find a basis for $range(T) \equiv \{\mathbf{y} \in \mathbb{R}^2 \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^2\}$. Note $\mathbf{A}_T$ row reduces to

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & -4 \end{bmatrix} \xrightarrow{R.R.E.F.} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$range(T) = ColSp(\mathbf{A}_T) = ColSp\left(\begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & -4 \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

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(b) Find a basis for $ker(T) \equiv \{\mathbf{x} \in \mathbb{R}^3 \mid T(\mathbf{x}) = \mathbf{0}\}$

$$ker(T) = NullSp(\mathbf{A}_T) = \text{solution set of } \left\{\begin{array}{rl} x_1 - 2x_3 & = 0 \\ 0 & = 0 \end{array}\right\} \implies \mathbf{x} = \left\{x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$= span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right)$$

4. (5 pts) Show that the set of functions that vanish at 0 is a subspace of the space of functions on the real line.
Let $\lambda \in R$ and f, g be functions such that $f(0) = 0 = g(0)$

$$(\lambda f)(0) = \lambda f(0) = \lambda \cdot 0 = 0$$
$$(f + g)(0) = f(0) + g(0) = 0 + 0$$

so the set of functions vanishing at 0 is closed under scalar multiplication and vector addition. So it is a subspace.
5. (10 pts) Find a basis for $W = span\left(1 + 2x - x^2, x - x^2, 2 + 3x - x^2\right) \subset P_2$.
Using the standard isomorphism between polynomials in $\mathcal{P}_2$ and vectors in $\mathbb{R}^3$: $a_0 + a_1x + a_2x^2 \longleftrightarrow [a_0, a_1, a_2]$, we find a basis for the span of the coordinate vectors of each of the polynomials $p_1 = 1 + 2x - x^2$, $p_2 = x - x^2$, $p_3 = 2 + 3x - x^2$

$$\begin{array}{rcl} p_1 = 1 + 2x - x^2 & \longleftrightarrow & [1, 2, -1] \equiv \mathbf{v}_1 \\ p_2 = x - x^2 & \longleftrightarrow & [0, 1, -1] \equiv \mathbf{v}_2 \\ p_3 = 2 + 3x - x^2 & \longleftrightarrow & [2, 3, -1] \equiv \mathbf{v}_3 \end{array} \implies \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = span([1, 0, 1], [0, 1, -1])$$

$$\implies span(p_1, p_2, p_3) = span\left(1 + x^2, x - x^2\right)$$

7. (10 pts) Let $\mathcal{P}$ be the vector space of polynomials in one variable. Show that

$$T : \mathcal{P} \to \mathcal{P} \quad , \quad p \to T(p) = \frac{dp}{dx}$$

is a linear transformation from $\mathcal{P}$ to $\mathcal{P}$.
Let $\lambda \in \mathbb{R}$, $p, q \in \mathcal{P}$

$$T(\lambda p) = \frac{d}{dx}(\lambda p) = \lambda \frac{dp}{dx} = \lambda T(p)$$

$$T(p+q) = \frac{d}{dx}(p+q) = \frac{dp}{dx} + \frac{dq}{dx} = T(p) + T(q)$$

Since T preserves both scalar multiplication and vector addition, it is a linear transformation.

8. Suppose V and W are vector spaces with bases, respectively, $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $B' = \{\mathbf{w}_1, \mathbf{w}_2\}$. Let $T : V \to W$ be a vector space homomorphism (linear transformation) such that

$$T(\mathbf{v}_1) = 2\mathbf{w}_1 - \mathbf{w}_2 \quad , \quad T(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2 \quad , \quad T(\mathbf{v}_3) = \mathbf{w}_2$$

(a) (5 pts) Find the matrix $\mathbf{A}_{T,B,B'}$ representing T (with respect to the bases B and B')

$$\mathbf{A}_{T,B,B'} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

(b) (5 pts) Suppose $\mathbf{v} \in V$ has coordinate vector $\mathbf{v}_B = [1, 2, 3]$. What is the coordinate vector of $T(\mathbf{v})$ with respect to the basis B'?.

$$(T(\mathbf{v}))_{B'} = \mathbf{A}_{T,B,B'}\mathbf{v}_B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

9. Calculate the following determinants (5 pts each).

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, determinant: -6

(b) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, determinant: 0

(c) $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$, determinant: -4