Math 3013
Solutions to Problem Set 5

1. Determine which of the following mappings are linear transformations.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_1 + x_2, x_1 - 3x_2]$

- This mapping is linear since if $\mathbf{v} = [x_1, x_2, x_3]$

$$\begin{aligned} T(\lambda\mathbf{v}) &= T(\lambda[x_1, x_2, x_3]) \\ &= T([\lambda x_1, \lambda x_2, \lambda x_3]) \\ &= [\lambda x_1 + \lambda x_2, \lambda x_1 - 3\lambda x_2] \\ &= \lambda[x_1 + x_2, x_1 - 3x_2] \\ &= \lambda T([x_1, x_2, x_3]) \\ &= \lambda T(\mathbf{v}) \quad (T \text{ preserves scalar multiplication}) \end{aligned}$$

and if $\mathbf{v} = [x_1, x_2, x_3]$ and $\mathbf{v}' = [x_1', x_2', x_3']$

$$\begin{aligned} T(\mathbf{v} + \mathbf{v}') &= T([x_1 + x_1', x_2 + x_2', x_3 + x_3']) \\ &= [x_1 + x_1' + x_2 + x_2', x_1 + x_1' - 3(x_2 + x_2')] \\ &= [x_1 + x_2, x_1 - 3x_2] + [x_1' + x_2', x_1' - 3x_2'] \\ &= T(\mathbf{v}) + T(\mathbf{v}') \quad (T \text{ preserves vector addition}) \end{aligned}$$

$\square$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^4 : T([x_1, x_2, x_3]) = [0, 0, 0, 0]$

- This mapping is linear since if $\mathbf{v} = [x_1, x_2, x_3]$

$$\begin{aligned} T(\lambda\mathbf{v}) &= T([\lambda x_1, \lambda x_2, \lambda x_3]) \\ &= [0, 0, 0, 0] \\ &= \lambda[0, 0, 0, 0] \\ &= \lambda T([x_1, x_2, x_3]) \\ &= \lambda T(\mathbf{v}) \quad (T \text{ preserves scalar multiplication}) \end{aligned}$$

and if $\mathbf{v} = [x_1, x_2, x_3]$ and $\mathbf{v}' = [x_1', x_2', x_3']$

$$\begin{aligned} T(\mathbf{v} + \mathbf{v}') &= T([x_1 + x_1', x_2 + x_2', x_3 + x_3']) \\ &= [0, 0, 0, 0] \\ &= [0, 0, 0, 0] + [0, 0, 0, 0] \\ &= T(\mathbf{v}) + T(\mathbf{v}') \quad (T \text{ preserves vector addition}) \end{aligned}$$

$\square$

(c) $T : \mathbb{R}^3 \to \mathbb{R}^4 : T([x_1, x_2, x_3]) = [1, 1, 1, 1]$

- This mapping is not linear since if $\mathbf{v} = [x_1, x_2, x_3]$

$$\begin{aligned} T(\mathbf{v}) &= [1, 1, 1, 1] \\ T(2\mathbf{v}) &= [1, 1, 1, 1] \neq 2[1, 1, 1, 1] = 2T(\mathbf{v}) \end{aligned}$$

So the mapping does not preserve scalar multiplication. $\square$

(d) $T : \mathbb{R}^2 \to \mathbb{R}^3 : T([x_1, x_2]) = [x_1 - x_2, x_2 + 1, 3x_1 - 2x_2]$

- This mapping is not linear since, e.g., if $\mathbf{v} = [1,1]$

$$\begin{aligned} T(\mathbf{v}) &= [0,2,1] \\ T(2\mathbf{v}) &= T([2,2]) = [0,3,2] \neq [0,4,2] = 2T(\mathbf{v}) \end{aligned}$$

So the mapping does not preserve scalar multiplication. □

2. For each of the following, assume T is a linear transformation, from the data given, compute the specified value.

(a) Given $T([1,0]) = [3,-1]$, and $T([0,1]) = [-2,5]$, find $T([4,-6])$.

- Because linear transformations preserve scalar multiplication and vector addition, they also preserve linear combinations:

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2)$$

Now take $\mathbf{e}_1 = [1,0]$ and $\mathbf{e}_2 = [0,1]$. Then

$$\begin{aligned} T([4,-6]) &= T(4\mathbf{e}_1 - 6\mathbf{e}_2) \\ &= 4T(\mathbf{e}_1) - 6T(\mathbf{e}_2) \\ &= 4[3,-1] - 6[-2,5] \\ &= [12+12, -4-30] \\ &= [24,-34] \end{aligned}$$

□

(b) Given $T([1,0,0]) = [3,1,2]$, $T([0,1,0]) = [2,-1,4]$, and $T([0,0,1]) = [6,0,1]$, find $T([2,-5,1])$.

- As in Part (a), we set $\mathbf{e}_1 = [1,0,0]$, $\mathbf{e}_2 = [0,1,0]$, and $\mathbf{e}_3 = [0,0,1]$ and then compute

$$\begin{aligned} T([2,-5,1]) &= T(2\mathbf{e}_1 - 5\mathbf{e}_2 + \mathbf{e}_3) \\ &= 2T(\mathbf{e}_1) - 5T(\mathbf{e}_2) + T(\mathbf{e}_3) \\ &= 2[3,1,2] - 5[2,-1,4] + [6,0,1] \\ &= [6-10+6, 2+5+0, 4-20+1] \\ &= [2,7,-15] \end{aligned}$$

□

3. Find the standard matrix representations of the following linear transformations.

(a) $T([x_1,x_2]) = [x_1+x_2, x_1-3x_2]$

- The standard matrix representations are computed by computing the action of the linear transformation T on the standard basis vectors, and then using results as the columns of the corresponding matrix. For the case at hand we have

$$\begin{aligned} \mathbf{e}_1 &= [1,0] \quad \Rightarrow \quad T(\mathbf{e}_1) = [1+0, 1-3(0)] = [1,1] \\ \mathbf{e}_2 &= [0,1] \quad \Rightarrow \quad T(\mathbf{e}_2) = [0+1, 0-3(1)] = [1,-3] \end{aligned}$$

So the matrix corresponding to T is

$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

□

(b) $T([x_1,x_2,x_3]) = [x_1+x_2+x_3, x_1+x_2, x_1]$

- We proceed as in Part (a).

$$\begin{array}{rcccl} \mathbf{e}_1 & = & [1,0,0] & \Rightarrow & T(\mathbf{e}_1) = [1+0+0, 1+0, 1] = [1,1,1] \\ \mathbf{e}_2 & = & [0,1,0] & \Rightarrow & T(\mathbf{e}_2) = [0+1+0, 0+1, 0] = [1,1,0] \\ \mathbf{e}_3 & = & [0,0,1] & \Rightarrow & T(\mathbf{e}_3) = [0+0+1, 0+0, 0] = [1,0,0] \end{array}$$

So the matrix corresponding to T is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

□

(c) $T : \mathbb{R}^3 \to \mathbb{R}^2 \; : \; T([x_1, x_2, x_3]) = [x_1 + x_2 + x_3, 2x_1 + 2x_2 + 2x_3]$

- We again compute the action of T on each standard basis vector $\mathbf{e}_i$ in the domain $\mathbb{R}^3$ of T and then use the results as the columns of $\mathbf{A}_T$:

$$\begin{array}{rcccl} \mathbf{e}_1 & = & [1,0,0] & \Rightarrow & T(\mathbf{e}_1) = [1+0+0\ ,\ 2+0+0] = [1,2] \\ \mathbf{e}_2 & = & [0,1,0] & \Rightarrow & T(\mathbf{e}_2) = [0+1+0, 0+2+0] = [1,2] \\ \mathbf{e}_3 & = & [0,0,1] & \Rightarrow & T(\mathbf{e}_3) = [0+0+1, 0+0+2, 0] = [1,2] \end{array}$$

$$\mathbf{A}_T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

4. For each of the linear transformations $T : \mathbb{R}^m \to \mathbb{R}^n$ in Problem 3, determine

$$Range(T) := \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^m\}$$

and

$$Kernel(T) := \{\mathbf{x} \in \mathbb{R}^m \mid T(\mathbf{x}) = \mathbf{0}_{\mathbb{R}^n}\} \quad .$$

- If the matrix corresponding to T is $\mathbf{A}_T$, then a basis for $Range(T)$ coincides with a basis for the $ColSp(\mathbf{A}_T)$ and a basis for $Kernel(T)$ coincides with a basis for $NullSp(\mathbf{A}_T)$. These two bases, in turn, can be calculated by reducing $\mathbf{A}_T$ to row echelon form and then interpreting that result accordingly. This I will do below for each of the matrices $\mathbf{A}_T$ computed in Problem 3.

  (a) Here we found

$$\mathbf{A}_T = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

This matrix row reduces to the following Reduced Row Echelon Form.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because there is a pivot in each column of the RREF, each column of $\mathbf{A}_T$ is a basis vector for $ColSp(\mathbf{A}_T) \approx Range(T)$. Thinking of $Range(T)$ as a subspace of $\mathbb{R}^2$, and writing vectors in $\mathbb{R}^2$ horizontally, we have

$$ColSp(\mathbf{A}_T) = span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right) \quad \Rightarrow \quad Range(T) = span([1,1], [1,-3])$$

Because there are no columns without pivots there are no free parameters in the solution of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$. Therefore, $\mathbf{x} = \mathbf{0}$ is the only solution and so $NullSp(\mathbf{A}_T) = \{\mathbf{0}\}$. Thus,

$$Kernel(T) = NullSp(\mathbf{A}_T) = \{\mathbf{0}\}$$

  (b) Here we found

$$\mathbf{A}_T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

This matrix row reduces to the following matrix in RREF

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Just like in part (a), there is a pivot in every column of the RREF and so each column of the original matrix is a basis vector for $ColSp(\mathbf{A}_T) \approx Range(T)$, Thus,

$$ColSp(\mathbf{A}_T) = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad \Rightarrow \quad Range(T) = span([1,1,1], [1,1,0], [1,0,0])$$

Also, there are no free parameters in the solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$, and so

$$Kernel(T) \approx NullSp(\mathbf{A}_T) = \{\mathbf{0}\}$$

(c) Here we found

$$\mathbf{A}_T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

This matrix row reduces to the following RREF

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since only the first column of the RREF contains a pivot, just the first column of $\mathbf{A}_T$ will provide a basis for $ColSp(\mathbf{A}_T) \approx Range(T)$. Thus,

$$ColSp(\mathbf{A}_T) = span\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \quad \Rightarrow \quad Range(T) = span([1,2])$$

There are two columns without pivots, thus two free parameters x_2,x_3 in the solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$. To get a basis for $NullSp(\mathbf{A}_T) \approx Kernel(T)$, we'll write down the general solution of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$ and grab the basis vectors from that. From the RREF form of $\mathbf{A}_T$ we get the following equations

$$\left.\begin{aligned} x_1 + x_2 + x_3 = 0 \\ 0 = 0 \end{aligned}\right\} \quad \Rightarrow \quad x_1 = -x_2 - x_3 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

And so

$$NullSp(\mathbf{A}_T) = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) \quad \Rightarrow \quad Kernel(T) = span([-1,1,0], [-1,0,1])$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

5. If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T([x_1,x_2]) = [2x_1 + x_2, x_1, x_1 - x_2]$ and $T' : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T'([x_1,x_2,x_3]) = [x_1 - x_2 + x_3, x_1 + x_2]$, find the standard matrix representation for the linear transformation $T' \circ T$ that carries $\mathbb{R}^2$ into $\mathbb{R}^2$. Find a formula for $(T' \circ T)([x_1,x_2])$.

- The matrix representations corresponding to T and T' are

$$\mathbf{M}_T = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \quad , \quad \mathbf{M}_{T'} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The matrix representation corresponding to $T' \circ T$ will be given by the product of the corresponding matrices

$$\mathbf{M}_{T' \circ T} = \mathbf{M}_{T'}\mathbf{M}_T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

Hence

$$(T' \circ T)(x_1, x_2) = [2x_1, 3x_1 + x_2]$$

: □