Math 3013
Solutions to Problem Set 6

1. Compute the determinants of the following matrices.

(a) $\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

- $\det \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = (1)(2) - (3)(-1) = 5$

(b) $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

- $\det \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix} = (1)(-1)^{1+1} \det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + (0)(-1)^{1+2} \det \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} + (-1)(-1)^{1+3} \det \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} =$
  $-1 + 0 + -5 = -6$

2. Use a cofactor expansion to compute the determinat of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 2 & 3 & 1 & 4 \end{bmatrix}$

- Cofactor expansions along the first row:

$$\begin{aligned}
\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 2 & 3 & 1 & 4 \end{pmatrix} &= (1)(-1)^{1+1} \det \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & 1 & 4 \end{pmatrix} + 0 + 0 + 0 \\
&= \det \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & 1 & 4 \end{pmatrix} = (2)(-1)^{1+1} \det \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix} \\
&= 2 \det \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix} \\
&= 2 \left((3)(-1)^{1+1} \det [4] + 0 \right) \\
&= (2)(3)(4) \\
&= 24
\end{aligned}$$

3. Use row reduction to compute the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & -1 & 3 \\ 2 & 1 & 1 & 2 \end{bmatrix}$

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\begin{aligned}
\det\left(\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & -1 & 3 \\ 2 & 1 & 1 & 2 \end{bmatrix}\right) &= \det\begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & -3 \\ 0 & -1 & 2 & -3 \\ 0 & -1 & 3 & -2 \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\
&= -\det\begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= -(1)(-1)(1)(0) \\
&= 0
\end{aligned}
$$

4. Use the result of Problem 1(b) to determine if the vectors $\mathbf{a} = [1, 0, -1]$, $\mathbf{b} = [3, 2, 1]$ and $\mathbf{c} = [-1, 1, 0]$ are linearly independent.

- If the vectors **a**, **b** and **c** are linearly independent, then the matrix **A** constructed by using **a**, **b** and **c** as rows (or columns) must have a *nonzero* determinant. The matrix in problem 1 (b) is has the vectors **a**, **b** and **c** as its rows and this matrix has determinant -6 which is not zero. Hence, the vectors **a**, **b** and **c** are linearly independent.

5. Use the result of Problem 3 to determine if the linear system

$$
\begin{aligned}
x_1 + x_2 - x_3 + 2x_4 &= 0 \\
2x_1 + x_2 + x_4 &= 0 \\
3x_1 + 2x_2 - x_3 + 3x_4 &= 0 \\
2x_1 + x_2 + x_3 + 2x_4 &= 0
\end{aligned}
$$

has a unique solution.

- An $n \times n$ linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution if and only if the determinant of the coefficient matrix is non-zero. The matrix in Problem 3 is the coefficient matrix for this linear system and its determinant is equal to 0. Therefore, this linear system does not have a unique solution.

6. Use Crammer's Rule to determine the solution (if any) of

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 2 \\
2x_1 - 2x_2 &= 4 \\
x_2 + x_3 &= 3
\end{aligned}
$$

- For this linear system we have

$$
\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad , \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}
$$

We form three associated matrices $\mathbf{B}_i$, $i = 1, 2, 3$, by replacing the i^{th} column of $\mathbf{A}$ with $\mathbf{b}$.

$$
\mathbf{B}_1 = \begin{pmatrix} 2 & 2 & 1 \\ 4 & -2 & 0 \\ 3 & 1 & 1 \end{pmatrix} \quad , \quad \mathbf{B}_2 = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 0 & 3 & 1 \end{pmatrix} \quad , \quad \mathbf{B}_3 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 4 \\ 0 & 1 & 3 \end{pmatrix}
$$

We have

$$\begin{aligned}\det(\mathbf{A}) &= -4\\ \det(\mathbf{B}_1) &= -2\\ \det(\mathbf{B}_2) &= 6\\ \det(\mathbf{B}_3) &= -18\end{aligned}$$

Crammer's Rule says the components x_1, x_2, x_3 of the solution vector are given by $x_i = \frac{\det(\mathbf{B}_i)}{\det(\mathbf{A})}$. Thus,

$$\begin{aligned}x_1 &= \frac{-2}{-4} = \frac{1}{2}\\ x_2 &= \frac{6}{-4} = -\frac{3}{2}\\ x_3 &= \frac{-18}{-4} = \frac{9}{2}\end{aligned}$$

7. Compute the cofactor $\mathbf{C}$ of matrix of $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and use the fomula

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T$$

to get a general formula for the inverse of a 2×2 matrix.

- First, we construct the cofactor matrix, $\mathbf{C}$. Its entries are given by

  $$(\mathbf{C})_{ij} = (-1)^{i+j} \det(\mathbf{M}_{ij})$$

  where $\mathbf{M}_{ij}$ is the matrix obtained from $\mathbf{A}$ by removing its i^{th} row and j^{th} column. Thus,

  $$\begin{aligned}C_{11} &= (-1)^{1+1}\det([d]) \quad , \quad C_{12} = (-1)^{1+2}\det([c])\\ C_{21} &= (-1)^{2+1}\det([b]) \quad , \quad C_{22} = (-1)^{2+2}\det([a])\end{aligned}$$

  or

  $$\mathbf{C} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

  We also have

  $$\det(\mathbf{A}) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

  and the transpose of $\mathbf{C}$ is

  $$\mathbf{C}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

  Thus,

  $$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$