Math 3013
Solutions to Problem Set 7

For each of the matrices $\mathbf{A}$ below, find

(a) the characteristic polynomial of $\mathbf{A}$
(b) the eigenvalues of $\mathbf{A}$
(c) for each real eigenvalue λ of $\mathbf{A}$, determine
  (i) the algebraic multiplicity of λ
  (ii) the geometric multiplicity of λ
  (iii) a basis for the λ-eigenspace of $\mathbf{A}$

1. $\mathbf{A} = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$

- We have

$$\begin{aligned} p_{\mathbf{A}}(\lambda) &= \det \begin{pmatrix} 7-\lambda & 5 \\ -10 & -8-\lambda \end{pmatrix} = (7-\lambda)(-8-\lambda) + 50 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda+3)(\lambda-2) \end{aligned}$$

We have two roots of $p_{\mathbf{A}}(\lambda) = 0$; $\lambda = -3$ and $\lambda = 2$. And so we have two eigenvalues for $\mathbf{A}$. Each of these eigenvalues has algebraic multiplicity 1, since there is only one factor of $(\lambda+3)$ in $p_{\mathbf{A}}(\lambda)$ and only one factor of $(\lambda-2)$ in $p_{\mathbf{A}}(\lambda)$.

We'll now find the corresponding eigenvectors.

$\lambda = -3$. We want to find a basis for the

$$NullSp(\mathbf{A} - (-3)\mathbf{I}) = NullSp\begin{pmatrix} 10 & 5 \\ -10 & -5 \end{pmatrix} = NullSp\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right)$$

Thus, a basis for the $\lambda = -3$ eigenspace will be $\mathbf{v}_{\lambda=-3} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ will be our eigenvector. Since the $\lambda = -3$ eigenspace is 1-dimensional, the geometric multiplicity of $\lambda = -3$ is 1.

$\lambda = 2$. Similarly, we have

$$NullSp(\mathbf{A} - (2)\mathbf{I}) = NullSp\begin{pmatrix} 5 & 5 \\ -10 & -10 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

Thus, a basis for the $\lambda = 2$ eigenspace will be $\mathbf{v}_{\lambda=2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Since the $\lambda = 2$ eigenspace is 1-dimensional, the geometric multiplicity of $\lambda = 2$ is 1.

2. $\mathbf{A} = \begin{bmatrix} -7 & -5 \\ 16 & 17 \end{bmatrix}$

- $$p_{\mathbf{A}}(\lambda) = \det\begin{pmatrix} -7-\lambda & -5 \\ 16 & 17-\lambda \end{pmatrix} = \lambda^2 - 10\lambda - 39 = (\lambda-13)(\lambda+3)$$

Thus, we have two eigenvalues $\lambda = 13$ and $\lambda = -3$, each with algebraic multiplicity 1.

$\lambda = 13$ eigenspace.

$$NullSp(\mathbf{A} - (13)\mathbf{I}) = NullSp\begin{pmatrix} -20 & -5 \\ 16 & 4 \end{pmatrix} = NullSp\begin{pmatrix} 1 & \frac{1}{4} \\ 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}\right)$$

Thus, a basis for the $\lambda = 13$ eigenspace will be $\mathbf{v}_{\lambda=13} = \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$. Since the $\lambda = 13$ eigenspace is 1-dimensional, the geometric multiplicity of $\lambda = 13$ is 1.

$\lambda = -3$ eigenspace.

$$NullSp(\mathbf{A} - (-3)\mathbf{I}) = NullSp\begin{pmatrix} -4 & -5 \\ 16 & 20 \end{pmatrix} = NullSp\begin{pmatrix} 1 & \frac{5}{4} \\ 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} -\frac{5}{4} \\ 1 \end{bmatrix}\right)$$

Thus, a basis for the $\lambda = -3$ eigenspace will be $\mathbf{v}_{\lambda=-3} = \begin{bmatrix} -\frac{5}{4} \\ 1 \end{bmatrix}$. Since the $\lambda = -3$ eigenspace is 1-dimensional, the geometric multiplicity of $\lambda = -3$ is 1.

3. $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$

- $$p_{\mathbf{A}}(\lambda) = \det \begin{pmatrix} 1-\lambda & -2 \\ 1 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) + 2 = \lambda^2 - 3\lambda + 4$$

  $p_{\mathbf{A}}(\lambda)$ is not easily factorizable; so we'll find its roots using the quadratic formula

  $$\lambda = \frac{-(-3) \pm \sqrt{(-3)^2 - (4)(1)(4)}}{(2)(1)} = \frac{3}{2} \pm \frac{\sqrt{-7}}{2} = \frac{3}{2} \pm \frac{\sqrt{7}}{2} i$$

  The eigenvalues are thus a pair of complex numbers (and the problem stops here since we have no real eigenvalues).

4. $\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix}$

- $$p_{\mathbf{A}}(\lambda) = \det \begin{pmatrix} -1-\lambda & 0 & 0 \\ -4 & 2-\lambda & -1 \\ 4 & 0 & 3-\lambda \end{pmatrix} = (-1-\lambda)(2-\lambda)(3-\lambda)$$

  Thus, we have three eigenvalues, $\lambda = -1$, $\lambda = 2$ and $\lambda = 3$. Each of these eigenvalues has algebraic multiplicity 1.

  $\lambda = -1$ eigenspace.

  $$NullSp(\mathbf{A} - (-1)\mathbf{I}) = NullSp \begin{pmatrix} 0 & 0 & 0 \\ -4 & 3 & -1 \\ 4 & 0 & 4 \end{pmatrix} = NullSp \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

  Thus, we are looking for a basis for the solution set of

  $$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

  Since the third column of the Reduced Row Echelon Form of $(\mathbf{A} - (-3)\mathbf{I})$ has no pivot, x_3 is a free parameter. The general solution vector is

  $$\mathbf{x} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \in span\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

  The vector $\mathbf{v}_{\lambda=-1} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ thus provides a basis for the $\lambda = -1$ eigenspace. Since the eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue -1 is also 1.

  $\lambda = 2$ eigenspace.

  $$NullSp(\mathbf{A} - (2)\mathbf{I}) = NullSp \begin{pmatrix} -3 & 0 & 0 \\ -4 & 0 & -1 \\ 4 & 0 & -1 \end{pmatrix} = NullSp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

(like we did for $\lambda = -1$, except, I'm only showing the major intermediary steps). The vector $\mathbf{v}_{\lambda=2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ thus provides a basis for the $\lambda = 2$ eigenspace. Since the eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue 2 is 1.

$\lambda = 3$ eigespace.

$$NullSp\left(\mathbf{A} - (3)\,\mathbf{I}\right) = NullSp\begin{pmatrix} -4 & 0 & 0 \\ -4 & -1 & -1 \\ 4 & 0 & 0 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$$

The vector $\mathbf{v}_{\lambda=3} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ thus provides a basis for the $\lambda = 3$ eigenspace. Since the eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue 3 is 1.

5. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -5 \\ 8 & 0 & 9 \end{bmatrix}$

- $$p_{\mathbf{A}}(\lambda) = \det\begin{pmatrix} 1-\lambda & 0 & 0 \\ -8 & 4-\lambda & -5 \\ 8 & 0 & 9-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda)(9-\lambda)$$

  We thus have three eigenvalues $\lambda = 1, \lambda = 4$ and $\lambda = 9$, and each of these eigenvalues has algebraic multiplicity 1.

  $\lambda = 1$ eigenspace.

$$NullSp\left(\mathbf{A} - (1)\,\mathbf{I}\right) = NullSp\begin{pmatrix} 0 & 0 & 0 \\ -8 & 3 & -5 \\ 8 & 0 & 8 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$$

The vector $\mathbf{v}_{\lambda=1} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ thus provides a basis for the $\lambda = 1$ eigenspace. Since the eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue 1 is 1.

$\lambda = 4$ eigenspace.

$$NullSp\left(\mathbf{A} - (1)\,\mathbf{I}\right) = NullSp\begin{pmatrix} -3 & 0 & 0 \\ -8 & 0 & -5 \\ 8 & 0 & -4 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

The vector $\mathbf{v}_{\lambda=4} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ thus provides a basis for the $\lambda = 4$ eigenspace. Since the eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue 4 is 1.

$\lambda = 9$ eigenspace.

$$NullSp\left(\mathbf{A} - (9)\,\mathbf{I}\right) = NullSp\begin{pmatrix} -8 & 0 & 0 \\ -8 & 5 & -5 \\ 8 & 0 & 0 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$

The vector $\mathbf{v}_{\lambda=9} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ thus provides a basis for the $\lambda = 9$ eigenspace. Since the eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue 9 is 1.

6. $\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ -7 & 2 & -1 \\ 7 & 0 & 2 \end{bmatrix}$

- $$p_{\mathbf{A}}(\lambda) = \det \begin{pmatrix} -4-\lambda & 0 & 0 \\ -7 & 2-\lambda & -1 \\ 7 & 0 & 2-\lambda \end{pmatrix} = (-4-\lambda)(2-\lambda)(2-\lambda) = (-4-\lambda)(2-\lambda)^2$$

We thus have 2 eigenvalues $\lambda = -4, 2$. The eigenvalue $\lambda = -4$ has algebraic multiplicity 1 (since there is 1 factor of $(-4-\lambda)$ in $p_{\mathbf{A}}(\lambda)$), while the eigenvalue $\lambda = 2$ has algebraic multiplicity 2 (since there are 2 factors of $(2-\lambda)$ in $p_{\mathbf{A}}(\lambda)$).

$\lambda = -4$ eigenspace.

$$NullSp(\mathbf{A} - (-4)\mathbf{I}) = NullSp \begin{pmatrix} 0 & 0 & 0 \\ -7 & 6 & -1 \\ 7 & 0 & 6 \end{pmatrix} = NullSp \begin{pmatrix} 1 & 0 & \frac{6}{7} \\ 0 & 1 & \frac{5}{6} \\ 0 & 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} -\frac{6}{7} \\ -\frac{5}{7} \\ 1 \end{bmatrix}\right)$$

The vector $\mathbf{v}_{\lambda=-4} = \begin{bmatrix} -\frac{6}{7} \\ -\frac{5}{7} \\ 1 \end{bmatrix}$ thus provides a basis for the $\lambda = -4$ eigenspace. Since the eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue -1 is 1.

$\lambda = 2$ eigenspace.

$$NullSp(\mathbf{A} - (2)\mathbf{I}) = NullSp \begin{pmatrix} -6 & 0 & 0 \\ -7 & 0 & -1 \\ 7 & 0 & 0 \end{pmatrix} = NullSp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

The vector $\mathbf{v}_{\lambda=2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ thus provides a basis for the $\lambda = 2$ eigenspace. Since the eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue 2 is 1.