Math 3013
Solutions to Problem Set 8

1. For each matrices below

- Determine if the matrix is diagonalizable.
- If the matrix is diagonalizable, find a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{C}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$.

(a) $\begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}$,

- We have

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (5-\lambda)(2-\lambda) + 2 = \lambda^2 - 7\lambda + 12 = (\lambda - 2)(\lambda - 4)$$

We thus have two distinct eigenvalues, $\lambda = 3, 4$. Since the algebraic multiplicity of each of these eigenvalues is 1, the matrix is diagonalizable. (This is because for each distinct eigenvalue we are guaranteed at least one linearly independent eigenvector, and we only need 2 linearly independent eigenvectors to compute the matrix $\mathbf{C}$.)

We'll now find the corresponding eigenvectors:

$\lambda = 2$:

$$\begin{aligned} Null(\mathbf{A} - (2)\mathbf{I}) &= NullSp\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} = NullSp\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \\ &= span\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = span\begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

So, we can use $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as the first column of $\mathbf{C}$.

$\lambda = 4$:

$$NullSp(\mathbf{A} - (6)\mathbf{I}) = NullSp\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = NullSp\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = span\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and so we can use $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the second column of $\mathbf{C}$. Thus,

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

The diagonal matrix $\mathbf{D}$ is formed by writing the corresponding eigenvalues of these eigenvectors in the same order along its main diagonal:

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

(b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

- We have

$$P_{\mathbf{A}}(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = \lambda^3 - \lambda^2 - 2\lambda = -\lambda(\lambda+1)(\lambda-2)$$

and so $\mathbf{A}$ has three distinct eigenvalues: $\lambda = 0, -1, 2$. Since each of its eigenvalues occurs with algebraic multiplicity 1, the matrix is diagonalizable.

$\lambda = 0:$

$$\begin{aligned} NullSp\left(\mathbf{A} - (0)\,\mathbf{I}\right) &= NullSp\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) \quad \Rightarrow \quad \mathbf{v}_{\lambda=0} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

$\lambda = -1$

$$\begin{aligned} NullSp\left(\mathbf{A} - (-1)\,\mathbf{I}\right) &= NullSp\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= span\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right) \Rightarrow \quad \mathbf{v}_{\lambda=-1} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

$\lambda = 2:$

$$\begin{aligned} NullSp\left(\mathbf{A} - (2)\,\mathbf{I}\right) &= NullSp\begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \\ &= span\left(\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right) = span\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_{\lambda=2} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

We thus have

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$, characteristic polynomial: $X^3 - 4X^2 + 5X - 2$

- We have

$$p_{\mathbf{A}}(\lambda) = \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 44 - \lambda \end{pmatrix} = \lambda^3 - 4\lambda^2 + 5\lambda - 2$$

: $(\lambda - 2)(\lambda - 1)^2$

Note that $\lambda = 1$ is a root of this polynomial; for

$$p_{\mathbf{A}}(1) = 1 - 4 + 5 - 2 = 0$$

Therefore, $(\lambda - 1)$ has to be a factor of $p_{\mathbf{A}}(\lambda)$. To complete the factorization of $p_{\mathbf{A}}(\lambda)$, we first divide it by $(\lambda - 1)$.

$$\frac{\lambda^3 - 4\lambda^2 + 5\lambda - 2}{\lambda - 1} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

and thus

$$p_{\mathbf{A}}(\lambda) = (\lambda - 1)^2 (\lambda - 2)$$

We thus have only two distinct eigenvalues $\lambda = 1, 2$.

- We need 3 linearly independent eigenvectors to construct an invertible matrix $\mathbf{C}$. We're guaranteed at least one linearly independent eigenectors with eigenvalue 2. However, it may happen that we also only get one linearly independent eigenvector for the eigenvalue $\lambda = 1$; so we'll look at this case first.

$\lambda = 1$:

$$NullSp\left(\mathbf{A} - (1)\,\mathbf{I}\right) = NullSp\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the reduced row echelon form only contains one column without a pivot (the 3^{rd} column), the null space of this matrix and hence, the $\lambda = 1$ eigenspace of the original matrix is only 1-dimensional. Thus, we'll only get one linearly independent eigenvector for the eigenvalue $\lambda = 1$. Since we need 3 linearly independent eigenvectors in order to construct an invertible matrix $\mathbf{C}$, and the other eigenvalue is only going to give us one more independent eigenvalue, we conclude that the original matrix is not diagonalizable.

2. Let $B = \{[1,-1],[1,1]\}$ and $B' = \{[1,2],[2,1]\}$

(a) Show B and B' are both bases for $\mathbb{R}^2$

- We can check for linear independence by writing the vectors in B an B' as the columns of a 2×2 matrix. If these matrices have nonzero determinants then both their rows and columns must be linearly independent.

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right) &= 1+1 \neq 0 \quad \Rightarrow \quad \text{the vectors in } B \text{ form a basis for } \mathbb{R}^2 \\ \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right) &= 1-4 \neq 0 \quad \Rightarrow \quad \text{the vectors in } B' \text{ form a basis for } \mathbb{R}^2 \end{aligned}$$

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(b) Find the change-of-basis matrix that converts vectors $[x_1, x_2]$ in $\mathbb{R}^2$ to their coordinate vectors with respect to the basis B.

- The matrix

$$\mathbf{C}_{B\to std} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

formed by using the vector of the basis B as columns will convert coordinate vectors with respect to B to standard vectors (i.e., coordinate vectors with respect to the standard basis $\{[1,0],[0,1]\}$). We want to go from standard vectors to coordinate vectors for B; thus, we need to compute the inverse of $\mathbf{C}_{B\to std}$:

$$\mathbf{C}_{std\to B} = \left(\mathbf{C}_{B\to std}\right)^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(c) Find the change-of-basis matrix that converts coordinate vectors with respect to B' to coordinate vectors with respect to B.

- We think of this change of basis as going from coordinate vectors for B' to the standard vectors and from standard vectors to coordinate vectors for B. Thus,

$$\mathbf{C}_{B'\to B} = \mathbf{C}_{std\to B}\mathbf{C}_{B'\to std} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$$

: and then from the standard basis to B'

3. Given that $B = \{[1,0,1],[0,1,1]],[0,0,1]\}$ and $B' = \{[1,1,-1],[1,1,0],[1,0,0]\}$ are two bases for $\mathbb{R}^3$. Find the change of basis matrix that converts coordinate vectors with respect to B' to coordinate vectors with respect to B.

- As in problem 2,We need to compute

$$\begin{aligned}
\mathbf{C}_{B'\to B} &= \mathbf{C}_{std\to B'}\mathbf{C}_{B'\to std} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -3 & -2 & -1 \end{pmatrix}
\end{aligned}$$

4. Find the change-of-basis matrix that converts vectors $[x_1, x_2]$ to their coordinate vectors with respect to a basis consisting of the eigenvectors of $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}$. (Hint: Use the results of problem 1 (a).)

- In problem 1(a) we say that $\mathbf{A}$ had two eigenvectors, $[1, 2]$ and $[1, 1]$, and in fact the matrix $\mathbf{C}$ that we formed by using the eigenvectors of $\mathbf{A}$ as columns, is interpretable as the change-of-coordinates matrix that takes coordinate vectors with respect to the basis $B = \{[1, 2], [1, 1]\}$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \mathbf{C}_{B\to std}$$

The desired change of basis matrix (from standard to B) is thus

$$\mathbf{C}_{std\to B} = (\mathbf{C}_{B\to std})^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

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