Math 3013
Solutions to Problem Set 9

1. Find a basis for the orthogonal complement $W^{\perp}$ of each of the subspaces given below¿

(a) $W = span\left([1,1,0],[1,0,1]\right)$ in $\mathbb{R}^3$

- First we use the given generators as the rows of a matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Any vector in the null space of this matrix will be perpendicular to the generators of W, and hence to every vector in W. So, in fact, $W^{\perp} = NullSp(\mathbf{A})$. We use row reduction to reduced row echelon form to determine a basis for $NullSp(\mathbf{A})$

$$NullSp\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} = span\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$$

So

$$W^{\perp} = span\left([-1,1,1]\right)$$

(b) $W =$ solution set of $x + y - z = 0$

- First, we need to identify a set of generators for W. W is the solution set of a the linear system $\mathbf{Ax} = 0$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$$

This simple matrix is already in reduced row echelon form, and its null space is

$$NullSp(\mathbf{A}) = span\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

Thus,

$$W = span\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

We now write the generators of W as the rows of a matrix and calculate its null space to get a basis for $W^{\perp}$.

$$NullSp\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = span\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$$

Thus,

$$W^{\perp} = span\left([1,1,-1]\right)$$

Note that we have just gone around full circle; $[1,1,-1]$ interpreted as a matrix is just the matrix $\mathbf{A}$ above. This is, in fact, because

$$\left(W^{\perp}\right)^{\perp} = W$$

2. For each vector $\mathbf{v}$ and subspace W below, determine the othogonal decomposition $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_{\perp}$ of $\mathbf{v}$ with respect to W.

(a) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$, $W = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

- We first find a basis for $W^{\perp}$.

$$W^{\perp} = NullSp\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix} = span\left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right)$$

We now set $B = \{[1,1,1], [1,-1,0], [1,1,-2]\}$. This is a basis for $\mathbb{R}^3$ where the first two basis vectors span W and the last basis vector spans $W^\perp$. We next find the change-of-basis matrix the converts standard vectors to coordinate vectors with respect to B

$$\mathbf{C}_{std\to B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

Thus,

$$\mathbf{v}_B = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ \frac{4}{3} \end{bmatrix}$$

And so

$$\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \left(\frac{2}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

:The part of $\mathbf{v}$ that lies in W is the sum of the first two terms

$$\mathbf{v}_W = \left(\frac{2}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

and the component of $\mathbf{v}$ that lies in $W^\perp$ is

$$\mathbf{v}_\perp = \begin{bmatrix} \frac{4}{3} \\ \frac{4}{3} \\ -\frac{8}{3} \end{bmatrix}$$

(b) $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix}$, $W = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right)$

- Here I'll use a more efficient way of determining $\mathbf{v}_w$ and $\mathbf{v}_\perp$. It goes as follows:
  - Find a basis $B_W = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ for W.
  - Find a basis $B_\perp = \{\mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ for $W^\perp$
  - Then finding the coordinate vector $\mathbf{v}_B = [x_1, \ldots, x_n]$ for $\mathbf{v}$ is equivalent to solving

    $$\mathbf{v} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n$$

    or

    $$\begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

    The latter matrix equation can be solved by row reducing the augmented matrix

    $$\left[\begin{array}{ccc|c} \mathbf{b}_1 & \cdots & \mathbf{b}_n & \mathbf{v} \end{array}\right]$$

    to its reduced row echelon form

    $$[\mathbf{I} \mid \mathbf{v}_B]$$

- In this problem we are given a basis for W, so the first step in the procedure above has already been done for us.

  We next need a basis for

$$W^\perp = NullSp\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} = span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right)$$

Now we set up the augmented matrix and row reduce:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -2 & 5 \\ 0 & 1 & -1 & 1 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 1 & 0 & -\frac{15}{2} \\ 0 & 0 & 0 & 1 & -6 \end{array}\right]$$

We can now write

$$\begin{aligned} \mathbf{v}_W &= (5)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{9}{2}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{19}{2} \\ 5 \\ \frac{1}{2} \\ \frac{9}{2} \end{bmatrix} \\ \mathbf{v}_\perp &= \left(-\frac{15}{2}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + (-6)\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2} \\ -6 \\ \frac{39}{2} \\ -\frac{27}{2} \end{bmatrix} \end{aligned}$$

3. For each set of linearly independent vectors below, apply the Gram-Schmidt process to obtain an orthogonal basis for the subspace generated by the vectors.

(a) $\left\{\begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$

- Let

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \quad , \quad \mathbf{b}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Then, following the Gram-Schmidt process we have

$$\mathbf{b}_1' = \mathbf{b}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{b}_2' &= \mathbf{b}_2 - \frac{\mathbf{b}_1' \cdot \mathbf{b}_2}{\mathbf{b}_1' \cdot \mathbf{b}_1'}\mathbf{b}_1' \\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{9-3}{9+9}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{aligned}$$

Thus

$$\left\{\begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}$$

is our new orthogonal basis

(b $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

-

$$\mathbf{b}_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{b}_2' = \mathbf{b}_2 - \frac{\mathbf{b}_1' \cdot \mathbf{b}_2}{\mathbf{b}_1' \cdot \mathbf{b}_1'}\mathbf{b}_1' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$\begin{aligned} \mathbf{b}_3' &= \mathbf{b}_3 - \frac{\mathbf{b}_1' \cdot \mathbf{b}_3}{\mathbf{b}_1' \cdot \mathbf{b}_1'}\mathbf{b}_1' - \frac{\mathbf{b}_2' \cdot \mathbf{b}_3}{\mathbf{b}_2' \cdot \mathbf{b}_2}\mathbf{b}_2' \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{3}}{\frac{1}{9}+\frac{1}{9}+\frac{4}{9}}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \end{aligned}$$

Thus,

$$B' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} , \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \right\}$$

is our new orthogonal basis.