Subsystems, Nilpotent Orbits, and Weyl Group Representations

OSU Lie Groups Seminar

November 18, 2009

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. (Everything I say will also be true for a complex reductive Lie algebra with obvious but otherwise inessential embellishments.) Let $\mathfrak{h}$ be some fixed Cartan subalgebra of $\mathfrak{g}$, $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, Π a choice of simple roots for Δ and Δ^+ the corresponding positive system. Throughout this talk, G will be the (complex) adjoint group of $\mathfrak{g}$ and W will be the Weyl group of $\mathfrak{g}$.

A subset Δ' is a subsystem of Δ if it is also a root system where the underlying vector space is $span_{\mathbb{R}}(\Delta') \subseteq \mathfrak{h}^*$ and the Euclidean inner product is just the restriction of the inner product on $\mathfrak{h}^*$ to vectors in $span_{\mathbb{R}}(\Delta')$. In the very near future, Huanrong is going to tell us how to explicitly identify all the subsystems of Δ up to W-conjugacy. Part of my agenda here today is to provide some motivations for such a classification. But perhaps first I should give some illustrative examples,

Example 1.1. Let Γ be a subset of the simple roots Π and set

$$W_\Gamma := \text{subgroup of } W \text{ generated by the reflections } s_\alpha,\ \alpha \in \Gamma \tag{1}$$

$$\Delta_\Gamma = W_\Gamma \cdot \Gamma = \{\alpha \in \Delta \mid w_\Gamma \beta \text{ for some } \alpha_1, \dots \alpha_k, \beta \in \Gamma\} \tag{2}$$

Then Δ_Γ is a subsystem of Δ. Moreover, Γ provides a simple base for Δ_Γ. Moreover

$$\mathfrak{s}_\Gamma = \langle \mathfrak{g}_\alpha \rangle_{\alpha \in \Delta_\Gamma} \equiv \text{ Lie subalgebra of } \mathfrak{g} \text{ generated by the root spaces } \mathfrak{g}_\alpha,\ \alpha \in \Delta_\Gamma. \tag{3}$$

is a semisimple subalgebra of $\mathfrak{g}$. It is in fact the semisimple part of the *standard Levi subalgebra* $\mathfrak{l}_\Gamma$

$$\mathfrak{l}_\Gamma := \mathfrak{h} \oplus \bigoplus_{\alpha \in span_{\mathbb{Z}}(\Gamma)} \mathfrak{g}_\alpha \tag{4}$$

In fact, every Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ is G-conjugate to some $\mathfrak{l}_\Gamma$, and moreover

$$G\text{-conjugacy classes of Levi subalgebras} \quad \xleftrightarrow{1:1} \quad W\text{-conjugacy classes of subsets of } \Pi \tag{5}$$

Naturally enough, we call subsystem Δ_Γ with $\Gamma \subset \Pi$, Levi subsystems. Unnaturally, though the convention is to call the corresponding Weyl group W_Γ a *parabolic subgroup* of W.

Example 1.2. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the simple roots and let α_0 be the lowest root with respect to the positive system defined by Π. The set

$$\Pi_e = \{\alpha_0, \alpha_1, \dots, \alpha_r\} = \Pi \cup \{\alpha_0\} \tag{6}$$

is called the *extended simple roots*. Like the simple roots themselves, the roots in Π_e are always mutually obtuse

$$\langle \alpha_i, \alpha_j \rangle \leq 0 \quad \text{whenever } i \neq j \tag{7}$$

However, the roots in Π_e are not linearly independent, so the "Dynkin diagram" formed from them will not be the Dynkin diagram of a semisimple Lie algebra. Yet, as there is only one dependence relation amongst the roots (viz., the canonical expression of α_0 as a negative-integer linear combination of the $\alpha_1, \dots, \alpha_k$), every proper subset of Π_e will consist of mutually orthogonal linearly independent roots, and so will correspond to the Dynkin diagram of a semisimple Lie algebra. In fact, if $\Gamma \subsetneq \Pi_e$, then we have

- a reflection subgroups W_Γ of W , via (1);
- a subsystem Δ_Γ of Δ, via (2);

- a semisimple subalgebra $\mathfrak{s}_\Gamma$ of $\mathfrak{g}$, via (3); and
- a reductivive subalgebra $\mathfrak{l}_\Gamma$ of $\mathfrak{g}$, via (4).

The reductive subalgebras $\mathfrak{l}_\Gamma$ in this case ($\Gamma \subsetneq \Pi_e$) are called *standard generalized Levi subalgebras*. More generally, a *generalized Levi subalgebra* is a subalgebra of $\mathfrak{g}$ that is G-conjugate to a standard generalized Levi subalgebras. Completing the analogy with Levi subalgebras we have

$$G\text{-conjugacy classes of generalized Levi subalgebras} \quad \overset{1:1}{\longleftrightarrow} \quad W\text{-conjugacy classes of subsets of } \Pi_e$$

Example 1.3. The generalized Levi subalgebras of generalized Levi subalgebras will again be reductive subalgebras of $\mathfrak{g}$. And so by iterating this process of spawning new subalgebras by taking subsets of the extended simple roots of a previously found subalgebras we can produce even more subsystem of Δ and corresponding reductive subalgebras of $\mathfrak{g}$. Of course, once you take a proper subset of Π_e and extend it by the lowest root of the subsystem it generates you end up leaving the confines of Π_e. But, nevertheless, following this algorithm, you still end up with subsets $\Gamma \subset \Delta$ with the properties

(8) $\langle \alpha, \beta \rangle \leq 0$ if $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$
(9) Γ is a linearly independent set.

and the formulas (1), (2), (3), (4) still yield respectively, a reflection subgroup W_Γ of W, a subsystem Δ_Γ of Δ, a semisimple subalgebra $\mathfrak{s}_\Gamma$ of $\mathfrak{g}$, and a reductive subalgebra $\mathfrak{l}_\Gamma$ of $\mathfrak{g}$. Eventually this algorithm, which is due to Borel and deSiebenthal, fails to produce any new Γ. The subalgebras $\mathfrak{l}_\Gamma$, or actually the set of G-conjugates of such subalgebras, so obtained, in fact, exhaust the set of reductive subalgebras of $\mathfrak{g}$ of maximal rank. Such subalgebras can be also characterized in another, very different, way: they correspond precisely to the Lie subalgebras that are left invariant by the action of a semisimple element of G. (N.B., these subalgebras are not, in general, the stabilizers of a semisimple element of $\mathfrak{g}$. This unseemly discrepancy is actually extremely useful, as such subalgebras are useful in detecting subtle properties of G-orbits in $\mathfrak{g}$).

Example 1.4. It is quite easy to see that the Weyl group of B_n is isomorphic to that of C_n. In fact, when we write use the standard presentations

$$\begin{aligned}\Delta(B_n) &= \{\pm(e_i \pm e_j) \mid i = 1, \ldots, n\} \cup \{\pm e_i \mid i = 1, \ldots, n\} \\ \Delta(C_n) &= \{\pm(e_i \pm e_j) \mid i = 1, \ldots, n\} \cup \{\pm 2e_i \mid i = 1, \ldots, n\}\end{aligned}$$

of the corresponding root system the isomorphism $W_{B_n} \longleftrightarrow W_{C_n}$ can be implemented by

$$s_{e_i} \longleftrightarrow s_{2e_i} \quad , \quad s_{e_i \pm e_j} \longleftrightarrow s_{e_i \pm e_j}$$

which amounts to the interchange of short and long roots. One can define a similar interchanges between the short and long roots of G_2 and F_4, by sending the long roots to themselves and rescaling the short roots by $\|\alpha_{long}\|^2 / \|\alpha_{short}\|^2$.

In each case, the root system obtained by rescaling the short simple roots by $\|\alpha_{long}\|^2 / \|\alpha_{short}\|^2$ produces a valid root system $\Delta^\vee$, the root system *dual* to Δ, to which there corresponds also a simple Lie algebra $\mathfrak{g}^\vee$, the *Lie algebra dual* of $\mathfrak{g}$. When $\mathfrak{g}$ is simply-laced (meaning all root lengths are the same) then we simply set $\Delta^\vee = \Delta$, and $\mathfrak{g}^\vee = \mathfrak{g}$.

Now let $\mathfrak{d}^\vee$ be the root system dual to Δ. One can apply the Borel-deSiebenthal algorithm to $\Delta^\vee$ to get a set of bases Γ for subsystems of $\Delta^\vee$. Applying the inverse duality map $\Delta^\vee \longrightarrow \Delta$, one obtains subsets $\Gamma \subset \Delta$ that retain the properties (8) and (9) and because of this allow one to attach to Γ a reflection subgroup W_Γ and a subsystem Δ_Γ of Δ.

However, the subalgebras $\mathfrak{s}_\Gamma$ arising from a $\Gamma^\vee \in \Delta^\vee$, in general, fail to close with in the confines of the root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Delta_\Gamma$. On the other hand, this extended Borel-deSiebenthall procedure does furnish one with (at least one W-conjugacy class representative of) all the subsystems of Δ and a representative of each conjugacy class of the Coxeter subgroups of W.

2. A Paradigm

Let me now return to the setting of Example 1, where Γ is a subset of Π. To Γ we can attach both a subsystem Δ_Γ and a Levi subalgebra $\mathfrak{l}_\Gamma$.

Using Δ_Γ we can also attach a particular irreducible representation of W, to Γ. This we do as follows. First of all, Γ actually provides a basis of simple roots for Δ_Γ, and so defines a positive system Δ_Γ^+. Set

$$p_\Gamma(x) = \prod_{\alpha \in \Delta_\Gamma^+} \langle \alpha, x \rangle \quad \in \quad \mathbb{C}[\mathfrak{h}] \tag{9}$$

Via results of Macdonald, p_Γ is a homogeneous W-harmonic polynomial on $\mathfrak{h}$ that generates an irreducible representation of W: i.e. W acts irreducibly on the span of W-translates of p_Γ. We denote by σ_Γ the corresponding representation of W.

Next, using $\mathfrak{l}_\Gamma$ we can attach a particular nilpotent orbit in $\mathfrak{g}$ to Γ. This is done as follows. Let $\mathfrak{p}_\Gamma = \mathfrak{l}_\Gamma + \mathfrak{u}$ be any extension of $\mathfrak{l}_\Gamma$ to a parabolic subalgebra of $\mathfrak{g}$. By well know results of Richardson, Lusztig and Spaltenstein, inside $G \cdot \mathfrak{u}$, there is a unique dense orbit $\mathcal{O}_{\mathfrak{l}_\Gamma}$ that is, in fact, independent of the particular parabolic extension used. This is the Richardson orbit corresponding to $\mathfrak{l}_\Gamma$. We shall denote it by $\mathcal{O}_\Gamma$.

We thus have the following picture,

$$\begin{array}{ccccc} \Gamma \subset \Pi & \longrightarrow & \mathfrak{l}_\Gamma & \longrightarrow & \mathcal{O}_\Gamma \\ \downarrow & & & & \\ \Delta_\Gamma & & & & \\ \downarrow & & & & \\ \sigma_\Gamma & & & & \end{array}$$

and so a natural, tight, correspondence between certain nilpotent orbits and certain representations of W.

3. The Springer Correspondence

Let $\mathcal{N}_\mathfrak{g}$ denote the (finite) set of nilpotent orbits of $\mathfrak{g}$. In a seminal 1978 paper, T.A. Springer gave a construction which attached to each irreducible representation of W, a particular G-equivariant local system on a particular nilpotent orbit $\mathcal{O} \in \mathcal{N}_\mathfrak{g}$..

Springer's construction goes as follows. Starting with a nilpotent element $X \in \mathfrak{g}$, let $\mathcal{B}_X$ be the variety of all Borel subalgebras of $\mathfrak{g}$ containing X. The stabilizer G^X of X in G obviously preserves $\mathcal{B}_X$, and this action in turn induces an action of G^X on the cohomology of $H^*(\mathcal{B}_X, \mathbb{C})$. Now the isotropy G^X is in general disconnected. It turns out that the identity component $\left(G^X\right)^o$ of G^Xacts trivially on $H^*(\mathcal{B}_X, \mathbb{C})$, and so the component group of G^X

$$A(X) \equiv G^X / \left(G^X\right)^o$$

has a well-defined action on $H^*(\mathcal{B}_X, \mathbb{C})$.

Springer showed that there is also a natural action of W on $H(\mathcal{B}_X, \mathbb{C})$, and this action commutes with that of $A(X)$. In fact, in the top degree cohomology $H^d(\mathcal{B}_X, \mathbb{C})$, where $d = \dim \mathcal{B}_X$, the representation of $A(X) \times W$ decomposes as

$$H^d(\mathcal{B}_X, \mathbb{C}) \approx \bigoplus_{\mu \in \widehat{A}(X)} m_\mu \chi_{\mu,X} \otimes \sigma_\mu$$

where the σ_μ are irreducible representations of W for which

$$\sigma_\mu \sim \sigma_\nu \quad \Longleftrightarrow \quad \chi_\mu \sim \chi_\nu$$

Moreover, let $\{X_i\}_{i \in I}$ be a complete set of representatives of the nilpotent orbits in $\mathfrak{g}$. Then

- Let $\{X_i\}_{i\in I}$ be a complete set of representatives of the nilpotent orbits of G. Then each $\sigma \in \widehat{W}$ appears as a $\sigma_\mu \in A(X_i)$ and for one and only one X_i and for one and only one $\mu \in \widehat{A}(X_i)$.
- For any given X_i, there trivial representation $1_{A(X_i)} \otimes \sigma_{1_{A(X_i)}}$ always occurs in the decomposition Not all representations σ

This last statement leads to the *Springer correspondence* that attaches to each nilpotent orbit X_i the unique representation .

$$s_{\mathcal{O}} : \mathcal{N} \hookrightarrow \widehat{W} : \mathcal{O}_X \longmapsto \sigma_{1_{A(X)}}$$

The first statement yields the *Springer parameterization* of $\widehat{W}$, in which each irreducible representation of W is uniquely specified by the orbit $\mathcal{O}_X$ and the particular $\mu \in \widehat{A}(X)$ for which $m_\mu \neq 0$.

Definition 3.1. *An irreducible representation σ of W is said to be an **orbit representation** if its Springer parameters are of the form $\left(\mathcal{O}, \mathbf{1}_{A(\mathcal{O})}\right)$. We denote by $\widehat{W}_{orbit}$ the set of orbit representations.*

The Springer correspondence is thus a bijective correspondence between $\mathcal{N}_{\mathfrak{g}}$ and $\widehat{W}_{orbit}$, albeit a rather artificial one if one has to rely on Definition 3.1 to identify $\widehat{W}_{orbit}$.

4. A Problem

The importance of the Springer correspondence to the developments in representation theory since the 1980's can not be overstated. One striking example is its use in the classification of the primitive ideals of $U(\mathfrak{g})$ (Joseph, Barbasch-Vogan). Another striking, but more subtle example, is the connection between Goldie rank polynomials, and wave-front expansions of characters of irreducible admissible representations is mitigated by the Springer correspondence (this was shown by D. King);.

Yet there is also something disturbingly awkward about the Springer parameterization, it as essentially an *a posteriori* parameterization of W. For one doesn't even know what Springer parameters are possible until one computes $H^d(\mathcal{B}_X, \mathbb{C})$ for each nilpotent orbit X. (In this respect, it is like the Dynkin-Kostant parameterization of nilpotent orbits via weighted Dynkin diagrams.)

On the other hand, in the paradigm discussed above, there was a very tight, in fact, a constructive correspondence between Richardson orbits $\mathcal{O}_\Gamma \subset \mathfrak{g}$ and the Macdonald representations $\sigma_\Gamma \in \widehat{W}$. This correspondence $\mathcal{O}_\Gamma \longleftrightarrow \sigma_\Gamma$, in fact, coincides with the restriction of the Springer correspondence to Richardson orbits. This circumstance hints at some sort of hitherto invisible naturality in the Springer correspondence (which we again point out is actually based on the identification of a certain family representations of W with Springer parameters of a similar form.).

So here we pose a problem:

Problem 4.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Can we find a common set of parameters $\mathcal{P}$ and constructions*

$$\begin{array}{ccccc} C_1 & : & \mathcal{P} & \xrightarrow{\ 1:1\ } & \mathcal{N}_{\mathfrak{g}} \\ C_2 & : & \mathcal{P} & \xrightarrow{\ 1:1\ } & \widehat{W}_{orbit} \end{array}$$

so that $C_2 \circ C_1^{-1}$ replicates the Springer correspondence?

Remark 4.1. For classical Lie algebras, both nilpotent orbits and Weyl group representations can be parameterized in terms of partitions. Moreover, there are algorithms for connecting the partition parameters of nilpotent orbits with the partition parameters of the corresponding orbit representations of W. However, these algorithms are different for each classical type, and are anyway unsuitable for handling the exceptional Lie algebras. So, while such algorithms don't really solve our problem in the desired generality, they do hint that a solution might exist.

5. Bala-Carter and Nilpotent Orbits

What makes the correspondence between Richardson orbits and the Macdonald representations so natural is the coincidence of the basic combinatorial datums Γ from which both the orbits and the representations are constructed. To prepare for a generalization of this correspondence, I will first recast the usual Bala-Carter classification of nilpotent orbits in combinational terms.

The basic idea of the Bala-Carter classification is the observation that any nilpotent element $X \in \mathfrak{g}$, is either a nilpotent element of a Levi subalgebra of $\mathfrak{g}$ or not. A nilpotent element that is not contained in any proper Levi subalgebra of $\mathfrak{g}$ is called a *distinguished nilpotent element*. It turns out that if a nilpotent element X of $\mathfrak{g}$ is not distinguished then there is a unique (up to G-conjugacy) Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ for which X is a distinguished element of $\mathfrak{l}$, and the set of nilpotent orbits of $\mathfrak{g}$ is in a $1:1$ correspondence with the set of G-conjugacy classes of pairs $(\mathfrak{l}, X)$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and X is a distinguished element of $\mathfrak{l}$.

We have seen in Example 1 above that the set of G-conjugacy classes of Levi subalgebras is in a $1:1$ correspondence with the set of W-conjugacy classes of subsets Γ of the simple roots of $\mathfrak{g}$. In fact, the subsets Γ correspond directly to the simple roots of the corresponding Levi subalgebra. In our combinatorial parameterization of nilpotent orbits we will simply use subsets of Π to parameterize Levi subalgebras.

What we need next is a combinatorial parameterization of the distinguished nilpotent orbits in a Levi subalgebra $\mathfrak{l}_\Gamma$. It turns out that the $Ad(\mathfrak{l}_\Gamma)$-orbit of a distinguished element of $\mathfrak{l}_\Gamma$ is always a Richardson orbit $\mathcal{R}_\gamma$ corresponding to some Levi subalgebra of $\mathfrak{l}_{\cdot\gamma}$ of $\mathfrak{l}_\Gamma$; where, naturally, $\gamma \subset \Gamma$. In fact, there is a very simple criterion for a given subset γ of Γ to produce a distinguished Richardson orbit in $\mathfrak{l}_\Gamma$:

Proposition 5.1. *The Richardson orbit*

$$\mathcal{R}_\gamma := ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_\Gamma}\left(\mathbf{0}_{\mathfrak{l}_\gamma}\right)$$

is distinguished in $\mathfrak{l}_\Gamma$ *if and only if*

$$|\Delta_\gamma| + |\Gamma| = \#\ \left\{\lambda \in \Delta_\Gamma^+ \mid \lambda = \alpha + \beta \quad ; \quad \alpha \in \Delta_\gamma\ ,\ \beta \in \Gamma \setminus \gamma\right\} \tag{10}$$

Combining this result with remarks above we have

Proposition 5.2. *The nilpotent orbits in* $\mathfrak{g}$ *are in a* $1:1$ *correspondence with the set of* W*-conjugacy classes of pairs* (Γ, γ) *where* Γ *is a subset of* Π *and* γ *is a subset of* Γ *satisfying (10).*

What is particularly nice about this parameterization is that is a constructive parameterization; which is to say, the orbit corresponding to (Γ, γ) can be constructed directly from its parameters: in fact, there is a remarkably succinct formula

$$\mathcal{O}_{(\Gamma,\gamma)} = inc_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_\Gamma}\left(\mathbf{0}_{\mathfrak{l}_\gamma}\right)\right)$$

Here

$$inc_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right) := \ G \cdot \mathcal{O}_{\mathfrak{l}} \equiv \{Z \in \mathfrak{g} \mid Z = g \cdot X' \text{ for some } g \in G \text{ and some } X' \in \mathcal{O}_{\mathfrak{l}}.\ \} \qquad .$$

and

$$ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right) := \text{the unique dense orbit in } G \cdot \left(\mathcal{O}_{\mathfrak{l}} + \mathfrak{u}\right)$$

where$\mathfrak{u}$ is the nilradical of any extension of $\mathfrak{l}$ to a parabolic subalgebra of $\mathfrak{g}$ (cf. the book by Collingwood-McGovern or the original paper by Lusztig and Spaltenstein).

Notation 5.3. We denote by $\mathcal{BC} = \mathcal{BC}_{\mathfrak{g}}$,any complete set of *combinatorial Bala-Carter parameters* for $\mathcal{N}_{\mathfrak{g}}$. That is to say, $\mathcal{BC}$ consists of pairs (Γ, γ) where $\Gamma \subset \Pi$ and γ is a subset of Γ satisfying (10); and the map

$$C : \mathcal{BC} \longrightarrow \mathcal{N}_{\mathfrak{g}} \qquad : \qquad (\Gamma, \gamma) \longmapsto inc_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_\Gamma}\left(\mathbf{0}_{\mathfrak{l}_\gamma}\right)\right)$$

is a bijection.

To construct a set of combinatorial Bala-Carter parameters, we just need to let the Γ's run through some complete set of representatives of $\Pi/\sim_W$ and then for each representative Γ, figure out the subsets $\gamma \in \Gamma$ that satisfy (10) and choose one representative from each W_Γ conjugacy class of such γ's.

6. The Barbasch-Vogan Duality Map

Having arrived at a nice constructive parameterization of nilpotent orbits in terms of certain pairs (Γ, γ), the next step in our program is construct from these same parameters, particular representations of W, in such a way that the correspondence between the orbits $\mathcal{O}_{(\Gamma,\gamma)}$ and the Weyl group representations $\sigma_{(\Gamma,\gamma)}$ is compatible with the Springer correspondence. Alas, we shall not achieve this. However, what we will accomplish is intrinsic characterization of the orbit representations of W in terms of Bala-Carter like parameters. This will be carried out by a natural extension of the Barbasch-Vogan duality map to Weyl group representations.

In their 1985 paper, Barbasch and Vogan define a map $\eta_{\mathfrak{g}}$ from the nilpotent orbits of a semisimple Lie algebra $\mathfrak{g}$ to the nilpotent orbits of the dual Lie algebra as follows. Let X be a nilpotent element in $\mathfrak{g}$. Via the Jacobson-Morozov theorem we can attach to x an S-triple $\{X, H, Y\}$. The semisimple element h of that S-triple can be reinterpreted first as an element of $(\mathfrak{h}^\vee)^*$, where $\mathfrak{h}^\vee$ is a Cartan subalgebra of the dual Lie algebra $\mathfrak{g}^\vee$, and then reinterpreted as corresponding to a particular infinitesimal character χ in $\widehat{Z(\mathfrak{g}^\vee)}$. The nilpotent orbit in $\mathfrak{g}^\vee$ corresponding to $X \in \mathfrak{g}$ is the associated variety of the maximal primitive ideal in $U(\mathfrak{g}^\vee)$ with infinitesimal character χ. We denote by $\eta_{\mathfrak{g}}(\mathcal{O}_X)$ the nilpotent orbit in $\mathfrak{g}^\vee$ obtained in this way from $\mathcal{O}_x = G \cdot X$.

The map $\eta_{\mathfrak{g}} : \mathcal{N}_{\mathfrak{g}} \longrightarrow \mathcal{N}_{\mathfrak{g}^\vee}$ has the following remarkable properties:

Theorem 6.1 (Barbasch-Vogan).

- *When $\mathfrak{g}$ is simply-laced, the map $\eta_{\mathfrak{g}}$ coincides with the Spaltenstein duality map, and with natural modifications replicates the Spaltenstein duality map on non-simply laced $\mathfrak{g}$. In particular, the image of $\eta_{\mathfrak{g}}$ is precisely the set of* ***special*** *nilpotent orbits in $\mathfrak{g}^\vee$.*
- *Suppose $\mathfrak{l}^\vee$ is a Levi subalgebra of $\mathfrak{g}^\vee$ dual to a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, and $\mathcal{O}_{\mathfrak{l}^\vee}$ is a nilpotent orbit in $\mathfrak{l}^\vee$. Then*

$$\eta_{\mathfrak{g}^\vee}\left(inc_{\mathfrak{l}^\vee}^{\mathfrak{g}^\vee}(\mathcal{O}_{\mathfrak{l}^\vee})\right) = ind_{\mathfrak{l}}^{\mathfrak{g}}(\eta_{\mathfrak{l}^\vee}(\mathcal{O}_{\mathfrak{l}^\vee})) \tag{12}$$

Observation 6.2. Suppose now that $\mathcal{O}_{\mathfrak{l}^{vv}} = \mathcal{R}_{\gamma^\vee}$ is a distinguished orbit in $\mathfrak{l}^\vee = \mathfrak{l}^\vee_{\Gamma^\vee}$. Then as we let the pairs $(\Gamma^\vee, \gamma^\vee)$ run over the set of Bala-Carter parameters for $\mathfrak{g}^\vee$, the left hand side of (12) runs over all the special nilpotent orbits in $\mathfrak{g}$. On the right hand, we have

$$ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(\eta_{\mathfrak{l}^\vee}\left(ind_{\mathfrak{l}_{\gamma^\vee}}^{\mathfrak{l}_{\Gamma^\vee}}(\mathbf{0})\right)\right)$$

Now each of the operations in this expression has a natural analog in the setting of Weyl group representations. The Richardson orbit $ind_{\mathfrak{l}_{\gamma^\vee}}^{l_{\Gamma^\vee}}(\mathbf{0})$ corresponds to the Macdonald representation of $W_{\Gamma^\vee}$ corresponding to the subsystem $\Delta_{\gamma^\vee}$ of $\Delta_{\Gamma^\vee}$. Let me denote this Macdonald representation by

$$j_{W_{\gamma^\vee}}^{W_{\Gamma^\vee}}(sgn(W_{\gamma^\vee}))$$

because Macdonald representations arise as special cases of a procedure known as truncated induction that is naturally analogous to induction of nilpotent orbits, and the sign representation of $W_{\gamma^\vee}$ is the representation of $W_{\gamma^\vee}$ that corresponds to the trivial nilpotent orbit of $\mathfrak{l}_{\gamma^\vee}$. We also replace the Barbasch-Vogan duality map by its W-analogy, the Lusztig duality map $\iota : \widehat{W} \longrightarrow \widehat{W}$, that sends σ to its twist by the sign representation of W (with a few adjustments for E_7 and E_8). One then arrives at a map

$$\Psi : \mathcal{BC}(\mathfrak{g}^\vee) \longrightarrow \widehat{W} \quad : (\Gamma^\vee, \gamma^\vee) \longmapsto j_{W_\Gamma}^{W}\left(\iota\left(j_{W_{\gamma^\vee}}^{W_{\Gamma^\vee}}(sgn(W_{\gamma^\vee}))\right)\right) \tag{13}$$

and one has

$$Image(\Psi) = \text{ the set of special representations of } W$$

That is the image of Ψ is exactly the set of representations of W that correspond to special nilpotent orbits via the Springer correspondence.

7. An Intermediary Result and a Conjecture

Now although we have introduced the special representations as those representations of W that correspond to special nilpotent orbits via the Springer correspondence, it should be pointed out that the notition of a special representation of W was originally defined in a completely W-intrinsic way [Lusztig]. Similarly, the notion of a special orbit was originally defined by allowing (a modification of) the partition-transpose operation to act on the partitions that parameterized the nilpotent orbits. The special orbits were defined as corresponding to the image of this action on the parameter space. In short, originally there was an orbit-centric notion of special orbits and a Weyl-group-centric notion of special representations and one of the first achievements of the Springer Correspondence was to provide a direct link between these two independent notions of special-ness.

In contrast, to this day, the notion of an orbit representation of W rests entirely on the Springer parameterization of W. The main result to be presented here is a completely W-centric characterization of orbit representations.

This will be quite easy, as basically all we have to do is expand the domain of the map Ψ (and then prove that the image of the new map coincides with the set of orbit representations).

We first note that the condition (10) for γ to be a distinguished subset of Γ still makes sense even if Γ is not a subset of Π. In particular, it makes sense when Γ is a subset of Π_e. It's just that the pair (Γ,γ) will no longer correspond to a nilpotent orbit (as there is no notion of orbit induction for generalized Levi subalgebras). On the other hand, the each of the objects and operations in (13) continue to make sense whenever Γ is a Coxeter base for a subsystem of Δ and γ is a subset of Γ.

In fact:

Theorem 7.1 (B–). *Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathcal{BC}_e(\mathfrak{g})$ denote the set of pairs (Γ,γ) where $\Gamma\subset\Pi_e$ and γ is a subset of Γ satisfying (10). Then the map*

$$\Psi_e : \mathcal{BC}_e(\mathfrak{g}^\vee) \longrightarrow \widehat{W} \quad : (\Gamma^\vee,\gamma^\vee) \longmapsto j_{W_\Gamma}^{W}\left(\iota\left(j_{W_{\gamma^\vee}}^{W_{\Gamma^\vee}}\left(sgn\left(W_{\gamma^\vee}\right)\right)\right)\right)$$

is well-defined and

$$image\left(\Psi_e\right) = \widehat{W}_{orbit}$$

This result allows one to separate the notion of orbit representation from the Springer parameterization. However, this map does not provide a parameterization of the orbit representations, it only locates them in $\widehat{W}$ in a completely W-centric way. On the other hand, having an W-intrinsic characterization of orbit representations may be enough to set up a W-intrinsic parameterization of the orbit representations. For the following seems to be true:

Conjecture 7.2 (B–). *Let $(\Gamma,\gamma)\in\mathcal{BC}$, the set of combinatorial Bala-Carter parameters for $\mathfrak{g}$.*

$$\phi_{\Gamma,\gamma} = \text{unique orbit representation occuring in } ind_{W_\Gamma}^{W}\left(j_{W_\gamma}^{W_G}\left(sgn\left(W_\gamma\right)\right)\right) \text{ of highest degree} \tag{14}$$

Then

$$\phi_{\gamma,\gamma} = Springer\left(\mathcal{O}_{(\Gamma,\gamma)}\right)$$

Remark 7.3. This is true for the exceptional Lie algebras. I haven't actually tried proving this for the classical case. Rather I have instead been tantalized by the prospect of understanding the right hand side of (14) as arising from some kind of cohomological construction of Weyl group representations You see, the induced module

$$M_{\Gamma,\gamma} = ind_{W_\Gamma}^{W}\left(j_{W_\gamma}^{W_G}\left(sgn\left(W_\gamma\right)\right)\right)$$

contains a whole lot of junk besides the desired orbit representation. There are lots of orbit representations of lower degree and lots of non-orbit representations of higher degree inside $M_{\Gamma,\gamma}$. The conjecture does provide a simple rule for singling out the desired orbit representation from all the junk, but in order to apply it you have to know all the orbit representations and their degrees. While we can get this information in a W-intrinsic way from the construction of Theorem 7.1, this still falls a bit short of what we really want: a direct construction of orbit representations from the Bala-Carter parameters of their corresponding orbits.

8. References

Subsystems
- N. Bourbaki, *Elements of Mathematics: Chapters 4-6, Lie Groups and Lie Algebras*, Springer, 1968.
- T. Oshima, *A Classification of Subsystems of a Root System* (preprint), `arXive:math/0611904.v4 [Math.RT]`

Subalgebras
- E.B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. Ser. 2, **6** (1957), 111-245.
- J. Tits, *Sous-algèbres des algèbres de Lie semi-simples*, Séminaire Bourbaki (Mai 1955), 119, 1-18.

W-reps
- R. Carter, *Simple Groups of Lie Type*, John Wiley and Sons, London, 1989.
- G. Lusztig, *A Class of irreducible representations of a Weyl group, II*, Indag. Math. **44** (1982), 219-226.
- G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, J. London Math. Soc. **19** (1979), 41-52.
- I.G. Macdonald, *Some irreducible representations of Weyl groups*, Bull. London Math. Soc., **4** (1972), 148-150.

Nilpotent Orbits
- P. Bala and R. Carter, *Classes of unipotent elements in simple algebraic groups, II*, Math. Proc. Camb. Philo. Soc. **80** (1976), 1-18.
- D. Collingwood and W. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold, New York, 1993.
- R.W. Richardson, *Conjugacy classes in parabolic subgroups of semisimple algebraic groups*, Bull. London Math. Soc. **6** (1974), 21-24.

Springer Corr.
- T.A. Springer, *A Construction of Representations of Weyl Groups*, Inv. Math. **44** (1978), 279-293.

Miscellaneous
- D. Barbasch and D. Vogan, *Unipotent representations of complex semisimple groups*, Ann. Math. **121** (1985), 41-110.
- A. Joseph, *Goldie rank in the enveloping algebra of a semisimple Lie algebra*, I,II, J. Alg. **65** (1980), 269-306.
- D. King, *The character polynomial of the annihilator of an irreducible Harish-Chandra module*, Amer. J. Math. **103** (1981), 1195-1240.