Lecture 4: LS Cells, Twisted Induction, and Duality

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Recap of Lecture 3

- $\mathfrak{g}$: a semisimple Lie algebra over $\mathbb{C}$
- $\Pi = \Pi(\mathfrak{h}; \mathfrak{g})$: a set of simple roots of Δ
- $\mathcal{N} = \mathcal{N}_{\mathfrak{g}}$: the cone of nilpotent elements in $\mathfrak{g}$

$$\mathcal{N} = \coprod_{(\Gamma,\gamma)} \mathcal{O}_{(\Gamma,\gamma)}$$

- $\Gamma \subset \Pi$: as *standard Gamma* representing a conjugacy class of Levi subalgebras
- $\gamma \subset \Gamma$: a *distinguished* subset of Γ, specifying a distinguished orbit $\mathcal{O}_\gamma = ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_\Gamma}(\mathbf{0})$ in $\mathfrak{l}_\Gamma$.

$$\mathcal{O}_{(\Gamma,\gamma)} = inc_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}\left(ind_{\mathfrak{l}_\gamma}^{\mathfrak{l}_\Gamma}(\mathbf{0})\right)$$

Γ standard $\equiv$ Γ is first in the lexicographic ordering of its W-conjugates

γ distinguished $\equiv$ γ has property

$$|\Delta_\gamma| + rnk\left([\mathfrak{l}_\Gamma, \mathfrak{l}_\Gamma]\right) = \#\left\{\alpha \in \Delta_\Gamma^+ \mid \alpha \text{ has exactly one simple component in } \Gamma - \gamma\ \right\}$$

Example: CBCP's for C_3

Gammas:

$\{\}$

$\{1\}$, $\{2\}$

$\{3\}$

$\{1,2\}$

$\{1,3\}$

$\{2,3\}$

$\{1,2,3\}$

$\rightarrow$ Standard Gammas $= \{\}$, $\{1\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$

Only $\Gamma = \{1,2,3\}$ has a non-trivial distinguished subset γ; viz, $\gamma = \{2\}$

C_3 Example cont'd

Positive roots (output of `posroots_rootbasis` command in atlas)

```
[0,2,1]
[0,1,1]
[0,1,0]
[0,0,1]
[1,2,1]
[1,1,1]
[1,1,0]
[1,0,0]
[2,2,1]
```

$\rightarrow \gamma = \{2\}$ is distinguished.

$\mathfrak{l}_\gamma = \mathfrak{sl}_2$

$|\Delta_\gamma| + rnk\left([\mathfrak{l}_\Gamma, \mathfrak{l}_\Gamma]\right) = 5$

CBCPs and partitions for $\mathfrak{sl}_n$

All Levis of $\mathfrak{sl}_n$ are sums of $\mathfrak{gl}_k$'s

For $\mathfrak{sl}(k)$ distinguished orbit $\Rightarrow$ principal orbit $\sim$ trivial γ

$\Rightarrow$ all CBCPs are of the form $[\Gamma, \{\}]$, with Γ some subset of $\{1, 2, 3, \ldots, n-1\}$.

Recipe for Partition:

- Form list $\mathbf{l} = \{\ell_1, \ell_2, \ldots, \ell_k\}$ of the lengths of the maximal strictly consecutive subsequences of Γ. E.g., in $\mathfrak{sl}_{15}$,

$$\Gamma = [1, 2, 3, 5, 6, 8, 9, 11, 13] \quad \Longrightarrow \mathbf{l} = [3, 2, 2, 1, 1]$$

- Add 1 to each entry in $\mathbf{l}$. E.g.,

$$[4, 3, 3, 2, 2]$$

- Add as many additional 1's to the tail of $\mathbf{l}$ as necessary to convert $\mathbf{l}$ into a partition of n. E.g.,

$$[4, 3, 3, 2, 2, 1] \in \mathcal{P}(15)$$

- The resulting partition of n is the partition corresponding to the orbit $\mathcal{O}_{[\Gamma, []]}$.

Partitions to CBCPs

Going back from partitions to Γ's is done by reversing previous recipe

- subtracting 1 from each part to get a list $\mathbf{l}$ of lengths of strictly consecutive subsequences of simple roots.
- Reconstruct a Γ from $\mathbf{l}$ in the obvious fashion.
(N.B. the resulting Γ will automatically be a *standard Gamma* for $\mathfrak{sl}(n)$.)

Ex.

$$\mathbf{p} = [3,2,2,1,1,1] \longrightarrow \mathbf{l} = [2,1,1,0,0,0] \longrightarrow \{1,2,4,6\}$$

$$CBCP = [\{1,2,4,6\}, \{\}]$$

CBCPs and partitions for $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(2n)$, and $\mathfrak{so}(2n)$

Definition

The **dismemberment** of a CBC diagram is the CBC obtained by removing all the circled nodes.
The G-**tail** of a CBC diagram is the connected component of the dismembered CBC diagram containing the last simple root of Π (w.r.t. Bourbaki ordering).
The A-**head** of a CBC diagram is what remains of the dismembered CBC diagram after the G-tail is removed.

Example

$$\bullet - \bullet - \circ - \bullet - \circ - \bullet - * \Longleftarrow \bullet \quad \begin{matrix} \nearrow \\ \searrow \end{matrix} \quad \begin{matrix} \bullet - * \Longleftarrow \bullet & G\text{-tail} \\ \\ \bullet - \bullet \quad \bullet & A\text{-head} \end{matrix}$$

N.B. The G-tail is the CBC diagram of a distinguished orbit of a simple factor of $\mathfrak{l}_\Gamma$ of the same Cartan type as $\mathfrak{g}$.

N.B. The A-head is consists of the Dynkin diagrams of the factors of $\mathfrak{l}_\Gamma$ that are of Cartan type A.

To determine the partition in $\mathcal{P}_G$ corresponding the orbit with CBCP $[\Gamma, \gamma]$

- Determine G-tail and A-head of CBC diagram for $[\Gamma, \gamma]$.
- The G-tail prescribes a Richardson orbit for a classical Lie algebra of the same Cartan type as $\mathfrak{g}$. Use the method described in lectures next week to compute the partition $\mathbf{p}_G$ corresponding to this Richardson orbit.
  (Or look up the G-tail partition in previously computed tables.)
- Construct the partition $\mathbf{p}_A$ corresponding to the A-head in exactly the same way as we did for Γ's for $\mathfrak{sl}(n)$.
- Concatenate $\mathbf{p}_G$ with **two** copies of $\mathbf{p}_A$ and then add as many 1's as necessary to get a partition in $\mathcal{P}_G$.
  N.B. the parity/multiplicity criteria of $\mathcal{P}_G$ will automatically be satisfied.
- The partition $\mathbf{p}_{[\Gamma,\gamma]}$ you end up with will be the partition in $\mathcal{P}_G$ corresponding to $\mathcal{O}_{[\Gamma,\gamma]}$

Example: a CBC diagram for $\mathfrak{sp}(16)$

$$\bullet - \bullet - \circ - \bullet - \circ - \bullet - * \Longleftarrow \bullet \quad \begin{array}{l} \nearrow \\ \searrow \end{array} \quad \begin{array}{ll} \bullet - * \Longleftarrow \bullet & G\text{-tail} \\ \\ \bullet - \bullet \quad \bullet & A\text{-head} \end{array}$$

$$\bullet - \bullet \quad \bullet \qquad \Longrightarrow \qquad [3,3,2,2]$$

$$\bullet - * \Longleftarrow \bullet \qquad \Longrightarrow \qquad [4,2] \quad (\text{from tables for } C_3)$$

$$\mathbf{p} = [4,3,3,2,2,2] \in \mathcal{P}_C(16)$$

Question: What is the largest Levi subalgebra containing a representative of the nilpotent orbit of $\mathfrak{sp}(16)$ corresponding to the partition $[4,3,3,2,2,2]$?

- We first try to view as $[4,3,3,2,2,2]$ as a concatenation of two partitions; one consisting of parts with even multiplicities and the other consisting of distinct even parts.

$$[4,3,3,2,2,2] \rightsquigarrow [3,3,2,2] \mid [4,2]$$

- The "subpart" $[3,3,2,2]$ corresponds to a A-head of the form $\bullet - \bullet - \circ - \bullet$.
- Consulting a table of distinguished orbits, one finds that the "subpart" $[4,2]$ is the partition attached to the distinguished orbit of $\mathfrak{sp}(6)$ with CBC diagram $\bullet - * \Leftarrow \bullet$. This is the G-tail of the CBC of $\mathcal{O}_{[4,3,3,2,2,2]}$.

- Attaching the A-head to the G-tail, with an empty node in between, we obtain the CBC diagram of $\mathcal{O}_{[4,3,3,2,2,2]}$.

$$\bullet - \bullet - \circ - \bullet - \circ - \bullet - * \Leftarrow \bullet$$

- Evidently, the maximal Levi subalgebra containing an $x \in \mathcal{O}_{[4,3,3,2,2,2]}$ will be of type $\mathfrak{gl}(3) + \mathfrak{gl}(2) + \mathfrak{sp}(3)$.
- Note that we get not only the isomorphism class of the desired Levi but also the simple roots generating its semisimple part.
- In fact, we see in **exactly** which distinguished orbit of the Levi subalgebra the element x resides.

The Plan for Today

Basic Theme: the organization of $G\backslash\mathcal{N}$

1. Closure Relations and Hasse Diagrams
2. Review of the Springer correspondence $G\backslash\mathcal{N} \hookrightarrow \widehat{W}$
3. Lusztig cells in $\widehat{W}$
4. Lusztig-Springer cells in $G\backslash\mathcal{N}$
5. Spaltenstein duality and Barbasch-Vogan duality
6. Twisted Induction and Intrinsic Duality

Closure Relations

Definition

Let $\mathcal{O}, \mathcal{O}' \in G\backslash\mathcal{N}$.

$$\mathcal{O} \leq \mathcal{O}' \quad \Longleftrightarrow \quad \mathcal{O} \subset \overline{\mathcal{O}'} \quad \text{(Zariski closure)}$$

Then $\leq$ is a partial ordering of the set $G\backslash\mathcal{N}$. It is called the **closure ordering** of $G\backslash\mathcal{N}$.

$G\backslash\mathcal{N}$ is thus a **poset** (partially ordered set)

Definition

The **covering relations** of a poset $(S, <)$ is the set

$$\mathcal{CR}(S, <) = \{(x, y) \in S \times S \mid x < y \text{ and } \nexists\, z \in S \text{ s.t. } x < z < y\}$$

The **Hasse diagram** of $(S, <)$ is directed graph whose vertices are the elements of S and whose directed edges (x, y) correspond to the pairs $(x, y) \in \mathcal{CR}(S, <)$.

Figuring Out Closure Relations

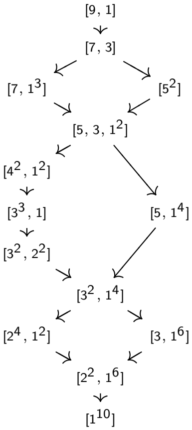
- If $\mathfrak{g}$ is of classical type:

$$\mathcal{O}_{\mathbf{p}} \leq \mathcal{O}_{\mathbf{q}} \quad \Longleftrightarrow \quad \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} q_j \quad ; \quad i = 1, \ldots, n$$

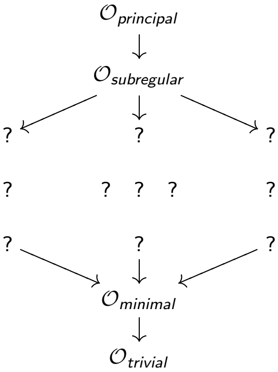
($\mathbf{p} = [p_1, \ldots, p_n]$, $\mathbf{q} = [q_1, \ldots, q_n]$; partitions in $\mathcal{P}_G$)

- If $\mathfrak{g}$ is of exceptional type, then the closure relations can be found in Spaltenstein, *Classes Unipotentes et Sous-Groups de Borel*, Lec. Notes in Math. **946** (1982).

Example: $\mathfrak{so}(10)$



In general:


$\mathcal{O}_{principal}$
$\mathcal{O}_{subregular}$
? ? ?
? ? ? ? ?
? ? ?
$\mathcal{O}_{minimal}$
$\mathcal{O}_{trivial}$

The Springer Correspondence

- $G\backslash\mathcal{N} \ni x \to u = exp(x) \in G$, unipotent element
- $\mathfrak{B}_u$: variety of all Borel subgroups of G that contain u.
- $H^i(\mathfrak{B}_u, \mathbb{Q})$: i^{th} cohomology group with coefficients in $\mathbb{Q}$
- Springer defines an action of the Weyl group W on the top cohomology group $V_u \equiv H^t(\mathfrak{B}_u, \mathbb{Q})$ $(t = \dim \mathfrak{B}_u)$
- $A(u) \equiv G^u/(G^u)_o$ also acts on V_u
- In fact, the W and $A(u)$ actions commute.
  So V_u decomposes as

$$V_u = \bigoplus_{\substack{\sigma \in \widehat{W} \\ \psi \in \widehat{A(u)}}} m_{\sigma,\psi} \sigma \otimes \psi$$

- Set

$$V_{u,\psi} \equiv \bigoplus_{\sigma \in \widehat{W}} m_{\sigma,\psi} \sigma \otimes \psi$$

Springer's Theorem

Theorem

(i) $V_{u,\psi}$, *when non-empty, is a direct sum of isomorphic irreducible* W*-modules: i.e.* $V_{u,\psi} = m\sigma \otimes \psi$ *for some* $\sigma \in \widehat{W}$.

(ii) *Let* $\sigma_{u,\psi}$ *be the irreducible representation of* W *corresponding to a* $V_{u,\psi} \neq 0$. *Then each irreducible representation of* W *occurs as a* $\sigma_{u,\psi}$ *for some unipotent element* $u \in G$ *and some irreducible character* ψ *of* $A(u)$.

(iii) $\sigma_{u,\psi} = \sigma_{u',\psi'}$ *if and only if* u *is conjugate to* u' *and* $\psi = \psi'$

(iv) *Thus, the irreducible representations of* W *are parameterized by pairs* $(G \cdot u, \psi)$ *where* $G \cdot u$ *is a unipotent conjugacy class of* G *and* ψ *is an irreducible character of* $A(u)$.

(v) $V_{u,1}$, *where* 1 *represents the trivial character of* $A(u)$, *is always non-zero.*

The last two statements are the basis of the Springer maps:

- $\widehat{W} \ni \sigma \longleftrightarrow (\mathcal{O}, \psi) \quad ; \quad \mathcal{O} \in G\backslash\mathcal{N} \quad , \quad \psi \in \widehat{A}(u)$ occuring in V_u
- $G\backslash\mathcal{N} \ni \mathcal{O} \mapsto \sigma_{u,1} \in \widehat{W}$

The last mapping is the Springer correspondence

$$s : G\backslash\mathcal{N} \hookrightarrow \widehat{W}$$

$\phi \in \widehat{W}$	$\mathcal{O}$	$A(u)$	$\chi \in \widehat{A}(u)$
$\phi_{1,0}$	G_2	1	1
$\phi_{2,1}$	$G_2(a_1)$	S_3	1
$\phi'_{1,3}$	$G_2\,(a_1)$	S_3	$\psi_{[2,1]}$
$\phi_{2,2}$	$\widetilde{A}_1$	1	1
$\phi''_{1,3}$	A_1	1	1
$\phi 1, 6$	$\mathbf{0}$	1	1

Notation: $\phi_{d,i}$ indicates an irreducible representation of W of dimension d occuring in $S(\mathfrak{h})$ in degree i but not occuring in $S(\mathfrak{h})$ in any homogeneous summand of lower degree. $\psi_{[2,1]}$ is the character of S_3 corresponding to the partition $[2,1]$.

Explicit Springer correspondences are tabulated in Carter's book *Finite Groups of Lie Type*.

Recall basic problem: How to organize the orbits in $G\backslash\mathcal{N}$?

In view of Springer correspondence, any organization of the reps in $\widehat{W}$ will induce a corresponding organization of orbits in $G\backslash\mathcal{N}$

Because of its strong connections with primitive ideal theory and parabolic induction, Lusztig's decomposition of $\widehat{W}$ into (left-) cells should be particularly interesting.

$\widehat{W}$ and $S(\mathfrak{h})$

Let W be a Weyl group regarded as a finite Coxeter group acting on $\mathfrak{h}^*$.
(A Coxeter group is a group generated by a set of reflections of a Euclidean space.)

Let $S(\mathfrak{h})$ be the symmetric algebra of $\mathfrak{h}$ identified as the ring of polynomial functions on $\mathfrak{h}^*$.

$$S(\mathfrak{h}) = \bigoplus_{i=0}^{\infty} S_i(\mathfrak{h}) \quad \text{(canonical grading by degree)}$$

Let $\overline{S}_i(\mathfrak{h})$ be the subspace of homogeneous W-harmonic polynomials of degree i. Then

$$W \cdot \overline{S}_i(\mathfrak{h}) \subset \overline{S}_i(\mathfrak{h})$$

Let

$$[\sigma : \overline{S}_i] \equiv \text{the multiplicity of } \sigma \text{ in } \overline{S}_i(\mathfrak{h})$$

Degree polynomials

Attached to each irreducible representation $\sigma \in \widehat{W}$ are two "degree" polynomials.

Definition

The **fake degree** polynomial P_σ of a irreducible representation $\sigma \in \widehat{W}$ is the polynomial

$$P_\sigma(X) = \sum_{i \geq 0} [\sigma : \overline{S}_i] X^i$$

The definition of the **generic degree** polynomial is much more technical.

$k = \overline{F_p}$: algebraic closure of a prime field

$\mathbb{G}$: adjoint Chevalley group over k with root system Δ.

$F_q \subset k$: finite field with q elements

G_q : the group of F_q rational points of $\mathbb{G}$

$h : \mathbb{C}[X]^* \to \mathbb{C}$ such that $h(X) = q$. ($\mathbb{C}[X]^*$ = integral closure of $\mathbb{C}[X]$)

Theorem (Benson-Curtis, 1972) h gives rise to a one-to-one correspondence between irreducible representations of W and irreducible representations of G_q occuring in $Ind_{B_q}^{G_q}(\mathbf{1})$.

$$\widehat{W} \ni \sigma \mapsto \psi_{\sigma,q} \in \widehat{G_q}$$

Moreover, $\dim \psi_{\sigma,q}$ is independent of choice of h and equals coincides with the value at $x = q$ of a certain well-defined polynomial $\widetilde{P}_\sigma(x)$ with rational q-independent coefficients.

Degree polynomials

Definition

The **fake degree polynomial** of $\sigma \in \widehat{W}$ is the polynomial $P_\sigma(X)$ given by

$$P_\sigma(X) = \sum_{i \geq 0} [\sigma : \overline{S}_i] X^i$$

The **generic degree polynomial** of $\sigma \in \widehat{W}$ is the polynomial $\widetilde{P}_\sigma(X)$ such that

$$\widetilde{P}_\sigma(q) = \dim \psi_{\sigma,q} \quad ; \quad \widehat{W} \ni \sigma \leftrightarrow \psi_{\sigma,q} \in Ind_{B_q}^{G_q}(\mathbf{1})$$

Definition

A representation $\sigma \in \widehat{W}$ is **special** if

$$\begin{aligned} P_\sigma(X) &= 1 \cdot X^a + \text{ terms of higher degree} \\ \widetilde{P}_\sigma(X) &= 1 \cdot X^a + \text{ terms of higher degree} \end{aligned}$$

Truncated Induction

Notation:
$\widetilde{a}_\sigma$: the lowest degree of X occuring in $\widetilde{P}_\sigma(X)$.

$W_\Gamma \subset W$: subgroup of W generated by the simple reflections corresponding to a subset Γ of the simple roots of $\mathfrak{g}$

Definition

Let σ' be an irreducible W_Γ-module.

$$J_{W_\Gamma}^{W}(\sigma') \equiv \sum_{\substack{\sigma \in \widehat{W} \\ \widetilde{a}_\sigma = \widetilde{a}_{\sigma'}}} \left[\sigma : Ind_{W_\Gamma}^{W}(\sigma')\right] \sigma$$

$J_{W_\Gamma}^{W}(\sigma')$ is the (in general, reducible) W-module obtained from σ' via **truncated induction**.

Remark: Truncated induction retains the usual transitivity property of ordinary induction (i.e., one can induce in stages) and it extends by linearity to the case when $\sigma' \in \widehat{W_\Gamma}$ is reducible.

We are finally in position to define Lusztig's **cell representations**.

Definition

If $W = \{e\}$, then there is only one **cell representation**, the unit representation of W. Assume now that $W \neq \{e\}$ and that for any $\Gamma \subset \Pi$, the cells of W_Γ have been defined. The **cell representations** of W are the (not necessarily irreducible) representations of W of the form $J_{W_\Gamma}^W(c)$ and those of the form $J_{W_\Gamma}^W(c) \otimes sgn(W)$, where Γ runs over the subsets of Π and c runs over the cell representations of W_Γ.

Definition/Construction: A cell rep is a W-rep realizable via truncated induction from a cell rep of a Levi subgroup W_Γ of W

$$J_{W_\Gamma}^{W}\left(\sigma'\right) \equiv \sum_{\substack{\sigma \in \widehat{W} \\ \widetilde{a}_\sigma = \widetilde{a}_{\sigma'}}} \left[\sigma : Ind_{W_\Gamma}^{W}\left(\sigma'\right)\right] \sigma$$

or by truncated induction followed by a twist by the sign rep of W

$$J_{W_\Gamma}^{W}\left(\sigma'\right) \otimes sgn$$

Basic Facts

- Every irreducible representation of W appears as a component of some cell representation.
- Every cell representation contains a unique special representation with multiplicity 1.
- Two cell representations have a common irrreducible component if and only if they have the same special component. ($\sigma \in \widehat{W}$ is **special** if the lowest order terms of the fake degree and generic degree polynomials agree.)

L-cells in $\widehat{W}$

Definition

Let σ be a special representation of W. The L-cell $\mathcal{C}_\sigma$ corresponding to σ is

$$\mathcal{C}_\sigma \equiv \left\{ \gamma \in \widehat{W} \mid \gamma \text{ is a constitutent of a cell rep containing } \sigma \right\}$$

Since every irreducible representation in $\widehat{W}$ belongs to some cell rep
and if σ in two different cell reps C, C' then C,C' share the same special component.
Thus,

$$\widehat{W} = \coprod_{\sigma \in \mathcal{S}} \mathcal{C}_\sigma$$

where $\mathcal{S}$ is the set of special representations of W and $\mathcal{C}_\sigma$ is the L-cell containing $\sigma \in \mathcal{S}$.

All the foregoing was set up the following definition.

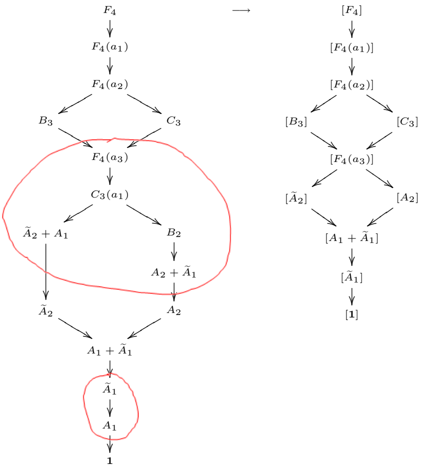
Definition

Let $s : G\backslash\mathcal{N} \hookrightarrow \widehat{W}$ be the Springer correspondence. A **Lusztig-Springer cell** is an equivalence class in $G\backslash\mathcal{N}$ under the equivalence relation

$$\mathcal{O} \sim \mathcal{O}' \quad \Longleftrightarrow \quad s(\mathcal{O}) \text{ and } s\left(\mathcal{O}'\right) \text{ belong to the same } L\text{-cell}$$

In Carter's book (*Simple Groups of Lie Type*) there are lists of the representations of W for the simple Lie algebras, separated into their various L-cells. Also in Carter's book are tables showing the Springer correspondence between nilpotent orbits and irreducible representations in W.

Nilpotent Orbits and LS-Cells for F_4



More generally

- Hasse diagrams for *LS*-cells are a bit simpler than that of nilpotent orbits
- For $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ partial ordering of *LS*-cells is actually a total ordering
- OTOH, for $\mathfrak{sl}_n$ every nilpotent orbit is special and resides by itself in an *LS*-cell. No simplication at all.
- Duality (next topic) is apparent in the Hasse diagram of *LS*-cells.

  (Duality manifests itself as a reflection symmetry of the Hasse diagram for *LS*-cells.)

Spaltenstein Duality: SL_n

Definition

Let $\mathbf{p} \in \mathcal{P}(n)$. The **transpose** of $\mathbf{p}$ is the partition $\mathbf{p}^t \in \mathcal{P}(n)$ defined by

$$\left(\mathbf{p}^t\right)_i = \#\left\{p_j \in \mathbf{p} \mid p_j \geq i\right\}$$

Example

$$[4,2,1] \sim \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array} \quad \longrightarrow \quad \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{array} \sim [3,2,1,1]$$

The tranpose operation $t : \mathcal{P}(n) \to \mathcal{P}(n)$ is an involution and induces an involution on the set of nilpotent orbits of SL_n.

Definition

Let $\mathcal{O}_{\mathbf{p}}$ be the nilpotent orbit of SL_n corresponding to a partition $\mathbf{p}(n) \in \mathcal{P}(n)$. The **Spaltenstein dual** of $\mathcal{O}_{\mathbf{p}}$ is the orbit corresponding to the partition $\mathbf{p}^t$.

Obstacle: Transpose operation does not preserve $\mathcal{P}_G(n)$ for other classical types.

Fact: Let $\mathbf{p} \in \mathcal{P}(N)$. Then there exists a unique maximal partition $\mathbf{p}_G \in \mathcal{P}_G$ that is dominated by $\mathbf{p}$. $\mathbf{p}_G$ is called the G-**collapse** of $\mathbf{p}$.

Example

The C-collapse of $[4, 3, 2, 1]$ is $[4, 2, 2, 2]$

$$[4,3,2,1] \sim \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \end{array} \longrightarrow \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & \bullet & & \\ \bullet & \bullet & & \end{array} \sim [4,2,2,2] \in \mathcal{P}_C$$

Definition

The **Spaltenstein dual** of a nilpotent orbit $\mathcal{O}_{\mathbf{p}}$, $\mathbf{p} \in \mathcal{P}_G$, of a classical Lie algebra is the orbit

$$d(\mathcal{O}_{\mathbf{p}}) = \mathcal{O}_{(\mathbf{p}^t)_G}$$

Example

$$[3,3,2,1,1] \sim \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{array} \rightarrow \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \end{array} \rightarrow \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \end{array} \sim [4,4,2] \in \mathcal{P}_C(10)$$

Special Orbits: Classical Groups

Theorem

Let G be a classical group.

$$d : G\backslash\mathcal{N} \to \mathcal{S} : \mathcal{O}_{\mathbf{p}} \mapsto \mathcal{O}_{(\mathbf{p}^t)_G}$$

restricts to an order-reversing involution on its image.

Definition

Let $\mathfrak{g}$ be a classical Lie algebra. A partition $\mathbf{p} \in \mathcal{P}_G$ is **special** if $\mathbf{p} = d(\mathbf{p}')$ for some $\mathbf{p}' \in \mathcal{P}_G$. Let $\mathcal{S}$ denote the corresponding set of **special nilpotent orbits**.

Facts

- Springer Correspondence maps special orbits to special representations of the Weyl group.
- Set of special orbits coincides with the set of associated varieties of primitive ideals of regular integral infinitesimal character

More generally

Spaltenstein defines a duality map for exceptional $\mathfrak{g}$ as well, but it is rather heuristic.

Barbasch-Vogan (1980) defined a canonical duality map $\eta_{\mathfrak{g}} : G\backslash\mathcal{N}_{\mathfrak{g}} \to G^{\vee}\backslash\mathcal{N}_{\mathfrak{g}^{\vee}}$

$$\eta_{\mathfrak{g}} : \mathcal{N}_{\mathfrak{g}} \ni x \to \{x, h, y\} \to \nu_h \in \left(\mathfrak{h}^{\vee}\right)^* \to I_{max} \in Prim\left(\mathfrak{g}^{\vee}\right)_{\nu_h} \to AV(I_{max}) \in \mathcal{N}^*_{\mathfrak{g}^{\vee}}$$

which, when combined with the correspondence $(\mathfrak{g}^{\vee})^* \leftrightarrow \mathfrak{g}^{\vee}$ arising from the Killing form and the Springer correspondences

$$\sigma_{\mathfrak{g}}^{-1} \circ \sigma_{\mathfrak{g}^{\vee}} : \left(G^{\vee}\backslash\mathcal{N}_{\mathfrak{g}^{\vee}}\right)_{special} \longrightarrow \widehat{W}_{special} \longrightarrow \left(G\backslash\mathcal{N}_{\mathfrak{g}}\right)_{special}$$

replicates the Spaltenstein duality map $d : G/\mathcal{N}_{\mathfrak{g}} \to \left(G/\mathcal{N}_{\mathfrak{g}}\right)_{special}$

David's homework assignment: There ought to be a canonical and intrinsically defined duality map $d : G\backslash\mathcal{N} \to \left(G\backslash\mathcal{N}\right)_{special}$.

Theorem (Barbasch-Vogan)

Suppose $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$, and $\mathfrak{l}^\vee$ is a Levi subalgebra of $\mathfrak{g}^\vee$ dual to $\mathfrak{l}$. If $\mathcal{O}_{\mathfrak{l}^\vee}$ is a nilpotent orbit in $\mathfrak{l}^\vee$ then

$$\eta_{\mathfrak{g}^\vee}\left(inc_{\mathfrak{l}^\vee}^{\mathfrak{g}^\vee}\left(\mathcal{O}_{\mathfrak{l}^\vee}\right)\right) = ind_{\mathfrak{l}}^{\mathfrak{g}}\left(\eta_{\mathfrak{l}^\vee}\left(\mathcal{O}_{\mathfrak{l}^\vee}\right)\right)$$

Observation: Every special orbit is induced from the dual of a distinguished orbit of a Levi subalgebra of $\mathfrak{g}^\vee$.

The Bala-Carter parameterization tells us that if we let $\mathfrak{l}^\vee$ run through the conjugacy classes of Levi subalgebras of $\mathfrak{g}^\vee$ while letting $\mathcal{O}_{\mathfrak{l}^\vee}$ run through the distinguished orbits of $\mathfrak{l}^\vee$, we hit every nilpotent orbit in $\mathfrak{g}^\vee$.

The special orbits in $G\backslash\mathcal{N}$ are precisely the orbits in the image of $\eta_{\mathfrak{g}^\vee}$.)

Intrinsic Duality

Definition

(Part 1) Suppose as an inductive hypothesis, we have defined the dual $d(\mathcal{O}_{\mathfrak{l}}) \in \mathcal{S}_{\mathfrak{l}}$ of any distinguished orbit $\mathcal{O}_{\mathfrak{l}}$ in a proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$.
Let $\mathcal{O}$ be any non-distinguished orbit in $\mathfrak{g}$ so that $x \in \mathcal{O}$ is a distinguished element of some proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$.
Then we define the (intrinsic) dual of $\mathcal{O}$ to be

$$d(\mathcal{O}) \equiv ind_{\mathfrak{l}}^{\mathfrak{g}}\left(d\left(\mathcal{O}_{\mathfrak{l}}\right)\right)$$

where $\mathcal{O}_{\mathfrak{l}}$ is the (unique) distinguished orbit in $\mathfrak{l}$ containing x.

Remarks:

A distinguished orbit is always a special orbit.

Theorem; (Spaltenstein) Any orbit induced from a special orbit is special.

So image of d (as so far defined) will always be a special orbit in $\mathfrak{g}$.

Facts about distinguished orbits

Every distinguished orbit is a Richardson orbit:

$$\mathcal{O} = ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0})$$

with Γ a distinguished subset of Π.

$\Rightarrow$ If $\mathcal{O}$ is distinguished in $\mathfrak{g}$, the set

$$C_{\mathcal{O}} \equiv \{(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}}) \mid \mathcal{O} = ind_{\mathfrak{l}}^{\mathfrak{g}}(d(\mathcal{O}_{\mathfrak{l}}))\}$$

is non-empty.

The dual of a principal orbit is always the trivial orbit.

If $\mathcal{O}_{\mathfrak{l},prin}$ is the principal orbit of the Levi subalgebra $\mathfrak{l}$ such that $\mathcal{O} = ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathbf{0})$, we have

$$ind_{\mathfrak{l}}^{\mathfrak{g}}(d(\mathcal{O}_{\mathfrak{l},prin})) = ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathbf{0}) = \mathcal{O}$$

So $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l},prin}) \in C_{\mathcal{O}}$

The set $\mathcal{C}_{\mathcal{O}}$

- The closure ordering of $G\backslash\mathcal{N}$ induces a partial ordering of

$$\mathcal{C}_{\mathcal{O}} \equiv \left\{ inc_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right) \mid \left(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}}\right) \in C_{\mathcal{O}} \right\}$$

  Think of $\mathcal{C}_{\mathcal{O}}$ as being the set of orbits whose Bala-Carter parameters lead to the same orbit $\mathcal{O}$ under the map

$$\left(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}}\right) \rightarrow ind_{\mathfrak{l}}^{\mathfrak{g}}\left(d\left(\mathcal{O}_{\mathfrak{l}}\right)\right)$$

- There is a unique maximal element of $\mathcal{C}_{\mathcal{O}}$, it is always a special orbit in the sense of Spaltenstein (empirical fact for exceptionals, provable by direct calculation in classical cases).

Definition

Let $\mathcal{O}$ be a distinguished orbit in $\mathcal{O}$. We define its (intrinsic) dual to be

$$d\left(\mathcal{O}\right) \equiv max\ \mathcal{C}_{\mathcal{O}}$$

Intrinsic Duality and Twisted Induction

Definition

We shall refer to the operation

$$\widetilde{ind}_{\mathfrak{l}}^{\mathfrak{g}} : L\backslash\mathcal{N}_{\mathfrak{l}} \to G\backslash\mathcal{N}_{\mathfrak{g}} \; : \; (\mathfrak{l}, \mathcal{O}_{\mathfrak{l}}) \mapsto ind_{\mathfrak{l}}^{\mathfrak{g}}\left(d\left(\mathcal{O}_{\mathfrak{l}}\right)\right)$$

as **twisted induction**

Definition

If $\mathcal{O}$ is a distinguished orbit in $\mathfrak{g}$, we define its dual orbit to be

$$d\left(\mathcal{O}\right) \equiv max\,\left\{inc_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right) \mid \mathcal{O} = \widetilde{ind}_{\mathfrak{l}}^{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{l}}\right)\right\}$$

More generally,

$$d(\mathcal{O}) \equiv ind_{\mathfrak{l}}^{\mathfrak{g}}\left(d\left(\mathcal{O}_{\mathfrak{l}}\right)\right)$$

where $\mathfrak{l}$ is a minimal Levi subalgebra containing a representative $x \in \mathcal{O}$ and $\mathcal{O}_{\mathfrak{l}}$ is the distinguished orbit in $\mathfrak{l}$ in which x resides.

Finale (Presto)

Definition

Let $\mathcal{O}$ be a special orbit in $G\backslash\mathcal{N}$ the corresponding **dual LS-cell** is the set

$$\mathcal{C}_{\mathcal{O}} = \left\{ \mathcal{O}_{(\mathfrak{l},\mathcal{O}_{\mathfrak{l}})} \mid \mathcal{O} = \widetilde{ind}_{\mathfrak{l}}^{\mathfrak{g}} \left(\mathcal{O}_{\mathfrak{l}}\right) \right\}$$

Theorem

- $$G\backslash\mathcal{N} = \coprod_{\mathcal{O}\in\mathcal{S}_{\mathfrak{g}}} \mathcal{C}_{\mathcal{O}}$$
- *The partitioning of $G\backslash\mathcal{N}$ arising from dual LS-cells coincides with that of the partitioning induced by Lusztig's cell decomposition of $\widehat{W}$ via the Springer correspondence.*

Somewhat anti-climactically, I point out that the *LS* cells are just the preimages of special orbits under the duality map.

The point, however, is that now the duality map is defined intrinsically, yet in a way that mimics the construction of cell representations.

Recycling Connections:

$$G\backslash\mathcal{N} \quad \begin{array}{c}\longrightarrow \\ \longleftarrow\end{array} \quad \widehat{W}$$

1. **Parameterize** sets
2. **Organize** sets by exploiting connections between sets
3. **Simplify**

Recycling Connections:

$$
\begin{array}{ccccc}
 & & \widehat{G}_{adm} & & \\
 & \nearrow\swarrow & \uparrow\downarrow & \searrow\nwarrow & \\
G\backslash\mathcal{N} & & \begin{array}{c}\longrightarrow \\ \longleftarrow\end{array} & & \widehat{W} \\
 & \searrow\nwarrow & \uparrow\downarrow & \nearrow\swarrow & \\
 & & Prim\,(U(\mathfrak{g})) & &
\end{array}
$$

1. **Parameterize** set
2. **Organize** exploiting connections between sets
3. **Simplify**